LONG TIME VALIDITY OF THE LINEARIZED BOLTZMANN UNCUT-OFF AND THE LINEARIZED LANDAU EQUATIONS FROM THE NEWTON LAW

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Abstract. We provide a rigorous justification of the linearized Boltzmann- and Landau equations from interacting particle systems with long-range interaction. The result shows that the fluctuations of Hamiltonian N- particle systems governed by truncated power law potentials of the form $\mathscr{U}(r) \sim |r/\varepsilon_{\text{eff}}|^{-s}$ (near $r \approx 0$) converge to solutions of kinetic equations in appropriate scaling limits $\varepsilon_{\text{eff}} \to 0$ and $N \to \infty$. We prove that for $s \in [0,1)$, the limiting system approaches the uncutoff linearized Boltzmann equation or the linearized Landau equation, depending on the scaling limit. The Coulomb singularity $s = 1$ appears as a threshold value. Kinetic scaling limits with $s \in (0,1]$ universally converge to the linearized Landau equation, and we prove the onset of the Coulomb logarithm for $s = 1$. To the best of our knowledge, this is the first result on the derivation of kinetic equations from interacting particle systems with long-range power-law interaction.

Contents

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1. Introduction

In kinetic theory, a gas of particles can be modelled by a system of N particles interacting *via* a potential $\mathscr{U}(\cdot/\varepsilon_{\text{eff}})$. In dimension 3, the power laws $\mathscr{U}_s(r) := r^{-s}$, $s \geq 1$ play a fundamental role, in particular the Coulomb case $s = 1$. One of the goals of kinetic theory is the description of such a gas in the limit $N \to \infty$, $\varepsilon_{\text{eff}} \to 0$. Of course, it depends on the scaling between ε_{eff} and N. Here we consider the low density scaling, where the occupied volume $N{\varepsilon_{\text{eff}}}^3$ goes to 0.

In the case $s > 1$, a first description of such a system is the Boltzmann equation: if at time 0 the particles are "sufficiently independent" (we do not precise the sens), the distribution of a typical particle $f(t, x, v)$ is a solution of the Boltzmann equation (introduced by Boltzmann in 1871 [Bol96])

$$\begin{gathered}\partial_t f + v \cdot \nabla_x f = Q_s(f, f)\\ Q_s(f,h)(v) := \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left(f(v')h(v'_*) - f(v)h(v_*)\right) b_s(v - v_*, \eta)\, \mathrm{d}v_*\, \mathrm{d}\eta,\\ v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\eta,\ v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\eta.\end{gathered} \tag{1.1}$$

where the kernel b_s depends on the potential $\mathscr{U}_s(\cdot)$. The collision operator Q can be interpreted as a jump operator for the velocities.

For a power law $\mathscr{U}_s$ with $s \geq 1$, the kernel b_s is equal to

$$b_s(z, \eta) = |z|^{\frac{s-4}{s}} q_s(z \cdot \eta),\ \text{with } q_s(\cos\theta) \underset{\theta \sim 0}{\sim} K\theta^{-\frac{2+s}{s}}$$

for some constant K. Hence the collision kernel is not integrable near the singularity $\eta \cdot \frac{v - v_*}{|v - v_*|} \simeq 1$ (when the collisions are grazing). We say that the Boltzmann kernel has no cut-off. However, the Boltzmann operator Q_s can be defined if functions f and h are differentiable.

In the Coulombian case $s = 1$, the singularity near is too large to defined the collision operator, even for smooth functions. In 1936, Landau proposed in [Lan36] an equation describing a low density Coulomb gas, the Landau equation:

$$\begin{gathered}\partial_t f + v \cdot \nabla_x f = cQ_L(f, f)\\ Q_L(f,h)(v) := \nabla_v \cdot \int_{\mathbb{R}^3} \frac{|v - v_*|^2 \mathrm{Id} - (v - v_*)^{\otimes 2}}{|v - v_*|^3} \big(\nabla f(v) h(v_*) - f(v) \nabla h(v_*)\big)\, \mathrm{d}v_*.\end{gathered} \tag{1.2}$$

The factor c a diffusion constant. The starting point of Landau's argument was the Boltzmann equation (1.1) associated with the cutoff collision kernel

$$b_{1,\alpha}(v - v_*, \eta) := \frac{1}{|\log \delta|} b_1(v - v_*, \eta) \mathbb{1}_{\left|\eta \cdot \frac{v - v_*}{|v - v_*|}\right| \leq 1 - \delta^2}.$$

The factor $\frac{1}{|\log \delta|}$ is the suitable normalisation sequence (sometimes called the *Coulomb logarithm*). In the limit $\delta \to 0$, the grazing collisions (collisions with small velocity jump) become dominant. The Landau collision operator is the limit of these Boltzmann operators. The rigorous justification of this process was performed by Alexandre and Villani in [AV04] (see also [AB91, Des92]).

One can ask if it is possible to derive the Boltzmann equation associated with $\mathscr{U}_s$ ($s < 1$), or the Landau equation, from a physical particle system. In fact, this question remains mainly open and the only results hold for compactly supported interaction potential and on short time intervals (look at the Section 1.1). However, it is possible to simplify the problem in order to obtain a positive answer.

A strategy for deriving the Boltzmann equation associated with the the potential $\mathscr{U}_s$ is to split the problem into two steps. First, we consider the truncated potential $\mathscr{U}_{s,R}(x) := \chi(R|x|)/|x|^s$, where $\chi(r) : \mathbb{R}^+ \to [0, 1]$ is a smooth, decreasing cut-off function:

$$\chi(0) = 1,\ \chi([1, \infty[) = \{0\},\ \chi' \leq 0.$$

Taking the Boltzmann-Grad limit $N \to \infty$, $N{\varepsilon_{\text{eff}}}^2 = 1$ ($N{\varepsilon_{\text{eff}}}^2 = (\log R)^{-1}$ in the Coulomb case $s = 1$), we want to recover the cut-off Boltzmann equation. In a second time, we take the grazing collision limit $R \to \infty$ to pass from the cut-off Boltzmann equation to the linearized Landau equation if $s = 1$

(respectively the linearized Boltzmann equation associated to $\mathscr{U}_s$ if $s > 1$). Assuming that R grows slower enough than N (we will need $R = O((\log\log N)^{1/4})$), one can take the two limits $N, R \to \infty$ simultaneously.

The difficulty is that we need the validity of the cut-off Boltzmann equation on a large time interval of order $O(1)$ (to be compared the validity time obtained by Lanford $O(1/R^2)$, [Lan75], see the next section). A good way to get long time results is to look at a system set initially near the thermodynamic equilibrium (or Gibbs state): the probability to find particles with position $x_1, \cdots, x_N$ and velocities $v_1, \cdots, v_N$ is

$$M^N_{\varepsilon_{\text{eff}},R}(x_1,\cdots,x_N,v_1,\cdots,v_N) := \frac{1}{\mathscr{Z}_{N,\varepsilon_{\text{eff}},R}} \exp\left(-\sum_{i=1}^N \frac{|v_i|^2}{2} - \sum_{1\leq i<j\leq N} \mathscr{U}_R\left(\frac{x_i - x_j}{\varepsilon_{\text{eff}}}\right)\right)$$

where $\mathscr{Z}_{N,\varepsilon_{\text{eff}},R}$ is a normalistion constant. We want to understand the fluctation field ζ^t_ε around the equilibrium: for a test function g, we define

$$\zeta^t_{\varepsilon_{\text{eff}}}(g) := \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N g(\mathbf{x}_i^{\varepsilon_{\text{eff}}}(t), \mathbf{v}_i^{\varepsilon_{\text{eff}}}(t)) - \mathbb{E}_{\varepsilon_{\text{eff}}}\left[\frac{1}{N}\sum_{i=1}^N g(\mathbf{x}_i^{\varepsilon_{\text{eff}}}(t), \mathbf{v}_i^{\varepsilon_{\text{eff}}}(t))\right]\right).$$

In the previous equality, $(\mathbf{x}_i^{\varepsilon_{\text{eff}}}(t), \mathbf{v}_i^{\varepsilon_{\text{eff}}}(t))$ denotes the coordinates of the i-th particle at time t, and the expectation is taken with respect to the Gibbs measure $M^N_{\varepsilon_{\text{eff}},R}\,\mathrm{d}X_N\,\mathrm{d}V_N$. In that setting, Bodineau *et al* are able to describe the hard sphere system by the linearized Boltzmann equation, which is derived considering a symmetric perturbation of the equilibrium (see [BGSR17, BGSRS21, LB22]).

One can now write a vague version of the theorem proved in the present paper (a rigorous version is written in Theorem 4).

Theorem 1. *Consider a system of N particles evolving with respect to Newton's laws, interacting through the pairwise potential $\mathscr{U}_{s,R}(\cdot/\varepsilon_{\text{eff}})$. At time zero, the particles are distributed with respect to the equilibrium measure $M^N_{\varepsilon_{\text{eff}},R}(x_1,\cdots,x_N,v_1,\cdots,v_N)\,\mathrm{d}X_N\,\mathrm{d}V_N$. Parameters $N, \varepsilon_{\text{eff}}, R$ are set with respect to the Boltzmann-Grad and grazing collision scallings*

$$N\to\infty,\ R\to\infty,\ R = O((\log\log N)^{1/4}),\ \text{and}\ N{\varepsilon_{\text{eff}}}^2 = \begin{cases} c_{s,\chi}R^{-2(1-s)} \text{ if } s\in(0,1)\\ (\log R)^{-1} \text{ if } s = 1,\\ 1 \text{ else.}\end{cases}$$

The diffusion constant $c_{s,\chi}$ (when $s \in (0,1)$) depends on the singularity s and the cut-off function χ.

Fix g and h two test functions. Then denoting $M(v) := \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$,

$$\mathbb{E}_\varepsilon\left[\zeta^t_{\varepsilon_{\text{eff}}}(g)\zeta^0_{\varepsilon_{\text{eff}}}(h)\right] \underset{\varepsilon_{\text{eff}}\to 0}{\longrightarrow} \int \mathbf{g}(t,x,v)h(t,x,v)M(v)\,\mathrm{d}x\,\mathrm{d}v \tag{1.3}$$

with $\mathbf{g}(t,x,v)$ the solution of the linearized equation

$$\begin{cases}\partial_t\mathbf{g} + v\cdot\nabla_v\mathbf{g} = \mathscr{L}_\infty\mathbf{g},\\ \mathbf{g}(t=0,x,v) = g(x,v)\end{cases} \text{where } \mathscr{L}_\infty\mathbf{g} := \begin{cases}\frac{1}{M}\left(Q_L(Mg,M) + Q_L(M,Mg)\right) \text{ if } s = 1,\\ \frac{1}{M}\left(Q_s(Mg,M) + Q_s(M,Mg)\right) \text{ else.}\end{cases}$$

1.1. **State of the art.** Now we recall some results about the derivation of the Boltzmann and Landau equation.

In the non linear setting, the only results hold for potential $\mathscr{U}(\cdot)$ supported in a ball $\{x\in\mathbb{R}^3,\ |x|\leq R\}$. In the Boltzmann-Grad scaling $N{\varepsilon_{\text{eff}}}^2 = 1$, the distribution of a typical particle follows the Boltzmann equation up to a time $O(1/R^2)$. The first derivation was performed by Lanford [Lan75] for hard spheres (*i.e.* $\exp(-\mathscr{U}_{\text{hs}}(r)) = \mathbb{1}_{r>1}$) and King [Kin75] for more general compactly supported potentials (see also [GSRT13, PSS14, Den18, BGSRS18]). The previous results have two defaults. They are valid only up to a small time (for the atmosphere at the level of the sea, this time scale is of order $10^{-9}s$), and the results apply only to a compactly supported interaction potential. However, there is one long time result out of equilibrium [IP89], in a setting where the dispersive effects are dominant.

For the Landau equation, the unique results hold only at time 0 (see [PSS14, Win21]): the authors obtain the equality

$$(\partial_t f)_{|t=0} = -v\cdot\nabla_x f_0 + Q_L(f_0,f_0).$$

This is not the first attempt to derive a linear version of the Boltzmann equation without a cut-off and the Landau equation. The first proof was provided in a setting where one tagged particle is followed in a background of fixed particles distributed with respect to the Poisson measure (see [DP99] for the derivation of Boltzmann equation and [DR01] for the derivation of Landau equation). Later, a *linear particles setting* was treated. One follows a tagged particle 1 in a bath initially at thermal equilibrium. Hence, the one has to compute the covariance

$$\mathbb{E}_{\varepsilon_{\text{eff}}}\left[h(\mathbf{x}_1^{\varepsilon_{\text{eff}}}(t),\mathbf{v}_1^{\varepsilon_{\text{eff}}}(t))g(\mathbf{x}_1^{\varepsilon_{\text{eff}}}(t),\mathbf{v}_1^{\varepsilon_{\text{eff}}}(t))\right]$$

in the Boltzmann-Grad and grazing collision scaaling. At the limit, the distribution of the particle 1 is described by the linear Landau (respectively Boltzmann) equation (see [Ayi17] for the Boltzmann version and [Cat] for the Landau one). In her paper, Ayi does not consider interaction through the cut-off potential $\mathscr{U}_{s,R}$, but directly a long range potential $\mathscr{U}$ with fast decay at infinity (she needs $\mathscr{U}(r)\leq O(\exp(-\exp\exp|x|^4)))$. Note that if the linear setting is a $O(1)$ perturbation of equilibrium, the linearized setting (which is treated in the present paper) is a $O(N)$ perturbation of equilibrium.

1.2. **Main steps of the proof.** We present now the main step of the proof. As explained before, the proof can be split into two pieces: first, we derive the linearized cut-off Boltzmann equation from the particle system, and second, we pass from the cut-off linearized Boltzmann equations to the linearized Landau equation (or the uncut-off Boltzmann). The second step has already been treated by Raphael Winter and the author [LBW22]. The main contribution of the present paper is the treatment of the first step, *i.e.* the derivation of the cut-off linearized Boltzmann equation.

This problem was already solved by Bodineau *et al* in the hard sphere setting [BGSRS21]. However, they need a refined result of Billard theory [BFK98] to control the dynamical memory effect (called recollision). It is an explicit bound on the number of collisions that can occur between a fixed number of particles. Such a result cannot be easily generalised to other interaction potentials, first because the "number of collision" is not well defined (particles can overlap). In [LB22], the author provided a proof avoiding the result of [BFK98]. It is based on a subtle conditioning of the initial data, allowing to control locally the number of recollisions. In the present work, we have simplified the strategy, and adapted it to particles interacting through a general compactly, supported interaction potential (see Assumption 2.1).

1.3. **Modification of the scaling parameters.** For a fix $s\geq 1$ and a cut-off function $\chi:\mathbb{R}^+\to[0,1]$, we define the parameters

$$\varepsilon:=\varepsilon_{\text{eff}}/R,\ \alpha:=1/R^s,$$

and the interaction potential

$$\mathscr{V}(x):=\frac{\chi(|x|)}{|x|^s}.$$

Hence we have the equality $\mathscr{U}_{s,R}(x/\varepsilon_{\text{eff}})=\alpha\mathscr{V}(x/\varepsilon)$.

In the following, we will use ε,α and $\mathscr{V}$ because it will simplify the notation and allows to take bounded potential $s=0$.

2. Definition of the system and strategy of the proof

2.1. **The Hamiltonian dynamic.** Let $\mathbb{T}:=\mathbb{R}^d/\mathbb{Z}^d$ (with $d\geq 2$) be the domain. We denote $\mathbb{D}=\mathbb{T}\times\mathbb{R}^d$ its tangent bundle and $\mathbb{D}^n$ the n-particle canonical phase space. In the following, we use the notation

$$X_n=(x_1,\cdots,x_n)\in\mathbb{T}^n,\ V_n=(v_1,\cdots,v_n)\in\mathbb{R}^{nd},\ \text{and}\ z_i=(x_i,v_i)\in\mathbb{D}.$$

On each $\mathbb{D}^n$ we construct the Hamiltonian dynamics associated with the Energy

$$\mathscr{H}_n(Z_n):=\frac{1}{2}|V_n|^2+\mathscr{V}_n(X_n),\quad \mathscr{V}_n(X_n):=\sum_{1\leq i<j\leq n}\alpha\mathscr{V}\left(\frac{|x_i-x_j|}{\varepsilon}\right),\tag{2.1}$$

where $\mathscr{V}$ is the interaction potential and α is a normalisation constant of the potential:

$$\forall i\in[1,n],\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}x_i=\nabla_{v_i}\mathscr{H}_n(Z_n(t))=v_i,\\ \frac{\mathrm{d}}{\mathrm{d}t}v_i=-\nabla_{x_i}\mathscr{H}_n(Z_n(t))=\dfrac{\alpha}{\varepsilon}\displaystyle\sum_{\substack{j=1\\ j\neq i}}^n\nabla\mathscr{V}\left(\frac{x_i-x_j}{\varepsilon}\right).\end{cases}\tag{2.2}$$

We impose the following condition on the interaction potential

Assumption 2.1. *There exists a constant* $s \in [0,\infty)$ *and a decreasing cut-off function* $\chi \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^2([0,1)$ *such that*

$$\mathscr{V}(x) := \frac{\chi(|x|)}{|x|^s},\ \chi(0)=1,\ \chi([1,\infty)) = \{0\}. \tag{2.3}$$

This dynamics is well defined for all times, almost everywhere in $\mathbb{D}^n$ with respect to the Lebesgue measure.

2.2. **Grand-canonical ensemble and stationary measure.** In the following, we choose to not fix the number of particles $\mathscr{N}$ but we define it as a random variable of expectation μ (we say that we take the *grand canonical ensemble*). If we choose the number of particles (the *Canonical ensemble*), the system will become more rigid and the calculations harder. However, one expects that Canonical and Grand Canonical ensembles become equivalent when the number of particles goes to infinity.

We denote $\mathscr{D} := \bigsqcup_{n\geq 0} \mathbb{D}^n$ the grand canonical phase space. We can then extend the Hamiltonian dynamics to $\mathscr{D}$ and denote $\mathbf{Z}_{\mathscr{N}}(t)$ the realisation (defined almost surely) of the Hamiltonian flow on $\mathscr{D}$ with random initial data $\mathbf{Z}_{\mathscr{N}}(0)$: for $\mathscr{N}=n$, $\mathbf{Z}_{\mathscr{N}}(t)$ follows the Hamiltonian dynamics on $\mathbb{D}^n$.

The initial data is sampled according to the stationary measure introduced now. The *grand canonical Gibbs measure* $\mathbb{P}_\varepsilon$ (and its expectation $\mathbb{E}_\varepsilon$) are defined on $\mathscr{D}$ as follows: an application $H:\mathscr{D}\to\mathbb{R}$ is a test function if there exists a sequence $(h_n)_{n\geq 0}$ with $h_n \in L^\infty(\mathbb{D}^n)$ and

$$\text{if } \mathscr{N}=n,\ \mathbf{Z}_{\mathscr{N}} = (\mathbf{z}_1,\cdots,\mathbf{z}_n) \text{ and } H(\mathbf{Z}_{\mathscr{N}}) := h_n(\mathbf{z}_1,\cdots,\mathbf{z}_n).$$

Then we define $\mathbb{E}_\varepsilon$ as

$$\mathbb{E}_\varepsilon[H(\mathbf{Z}_{\mathscr{N}})] := \frac{1}{\mathscr{Z}} \sum_{n\geq 0} \frac{\mu^n}{n!} \int_{\mathbb{D}^n} h_n(Z_n)\, \frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{nd/2}}\, \mathrm{d}Z_n, \tag{2.4}$$

where $\mathscr{Z}$ is a normalisation constant and μ is the chemical potential. The parameters ε and μ are tuned with respect to the Boltzmann-Grad scaling

$$\mu\,\varepsilon^{d-1}\mathfrak{d} = 1, \tag{2.5}$$

where $\mathfrak{d}$ is the *mean free path.* The length $\mathfrak{d}$ can be interpreted as the typical distance crossed by a particle between two collisions.

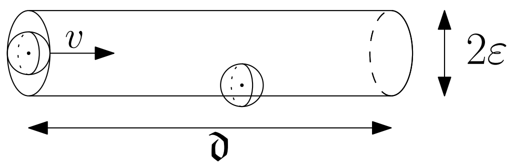

FIGURE 1. The first particle will meet the second one. Here v is of order 1.

The empirical distribution at time t is defined as the average configuration of particles at time t: for any g test function on $\mathbb{D}$,

$$\pi_t^\varepsilon(g) := \frac{1}{\mu} \sum_{i=1}^{\mathscr{N}} g(\mathbf{z}_i(t)). \tag{2.6}$$

At equilibrium, we have the following law of large numbers. Denote

$$M(v) := \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{d}{2}}}. \tag{2.7}$$

Proposition 2.2. *For any continuous and bounded test function* $g:\mathbb{T}\times\mathbb{R}^d\to\mathbb{R}$*, for all* $t\in\mathbb{R}$ *and for any* $\delta>0$*,*

$$\lim_{\varepsilon\to 0} \mathbb{P}_\varepsilon\left[\left|\pi_\varepsilon^t(g) - \int g(z)M(v)\,\mathrm{d}z\right| \geq \delta\right] = 0. \tag{2.8}$$

Remark 2.2.1. *The previous result is a simple corollary of the Lanford theorem and of the invariance of the measure (see* [Lan75]*).*

The aim of this article is to investigate the next order, namely the *fluctuation field*

$$\zeta_\varepsilon^t(g) := \mu^{\frac{1}{2}}\left(\frac{1}{\mu}\sum_{i=1}^{\mathcal{N}} g(\mathbf{z}_i(t)) - \mathbb{E}_\varepsilon[\pi_0^\varepsilon(g)]\right). \tag{2.9}$$

2.3. **Binary collision, scattering and definition of the Linearized Boltzmann operator.** Interactions involving more than two particles become negligible in the Boltzmann-Grad limit.

The present section is dedicated to describing the map between pre-collision and post-collision velocities. It is called the *scattering map.*

Consider two interacting particles 1 and 2 following the Hamiltonian dynamic associated with $\mathscr{H}_2$. At time 0, particles have coordinates $(X_2(0), V_2(0))$ with

$$x_1(0) = \varepsilon\nu,\ x_2(0) = 0,\ v_1 = v \text{ and } v_2(0) = v_*$$

where $\nu \in \mathbb{S}^{d-1}$ and $(v - v_*)\cdot\nu > 0$.

The particles will interact on a finite interval $[0, [\tau]]$ with τ the infimum of $\{\tau > 0,\ |x_2([\tau]) - x_1([\tau])| = \varepsilon\}$. The time $[\tau]$ is finite and bounded by $\frac{1}{|v-v_*|}$ (see Lemma B.1). We define (ν', v', v'_*) as

$$\nu' := \frac{x_2(\tau) - x_1(\tau)}{\varepsilon} \text{ and } (v', v'_*) := \big(v_1([\tau]), v_2([\tau])\big).$$

In addition, it conserves both momentum, kinetic energy, and angular momentum:

$$v + v_* = v' + v'_*,\ |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 \text{ and } (v - v_*)\wedge\nu = (v' - v'_*)\wedge\nu', \tag{2.10}$$

with $\wedge$ the cross product. We deduce that

$$|(v - v_*)\cdot\nu| = |(v' - v'_*)\cdot\nu'|. \tag{2.11}$$

Definition 2.2.1. *The* scattering application

$$\xi_\alpha : (\nu, v, v_*) \mapsto (\nu', v', v'_*) \tag{2.12}$$

is a local diffeomorphism which sends measure $\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\nu$ *to* $\mathrm{d}v'\,\mathrm{d}v'_*\,\mathrm{d}\nu'$.

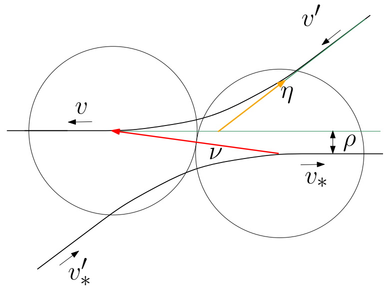

FIGURE 2. The scattering between two particles.

We define the linearized Bolzmann operator in the King's form:

$$\mathscr{L}_{\mathscr{U}}\,g(v) := \int_{\mathbb{S}\times\mathbb{R}^d} \big(g(v') + g(v'_*) - g(v) - g(v_*)\big)((v - v_*)\cdot\nu)_+ M(v_*)\,\mathrm{d}\nu\ \mathrm{d}v_*. \tag{2.13}$$

where we apply the scattering with interaction potential $\mathscr{U}(\cdot)$, and $\mathscr{L}_\alpha := \mathscr{L}_{\alpha\mathscr{V}}$.

This operator describes the variation of mass due to changes of velocity of colliding particles. The operator $\mathscr{L}_\alpha$ is a self-adjoint non-positive operator on $L^2(M(v)\,\mathrm{d}z)$.

Remark 2.2.2. *We say that the Boltzmann operator* $\mathscr{L}_{\mathscr{U}}$ *has a cutoff because we cutoff the long range interaction.*

There is another interpretation of this property. For parameter (v, v_*, ν), *we can define the vector* η *such that*

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\eta,\ v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\eta,$$

and $b_\alpha(v - v_*, \eta)$ *(called the* <u>collision kernel</u>*) the Jacobian of the application* $\nu \mapsto \eta$*:*

$$((v - v_*)\cdot\nu)_+\,\mathrm{d}\nu \to b_\alpha(v - v_*, \eta)\eta.$$

We say that the Boltzmann operator has a cutoff because for any $v - v_*$, the following bound holds

$$\int b_\alpha(v - v_*, \eta)\, \mathrm{d}\eta < \infty.$$

2.4. **Convergence to the linearized Boltzmann equation with a cut-off.** We recall that we have divided the proof of Theorem 1 into two steps. The first step is to take the Boltzmann-Grad limit $\mu \to \infty$. As we want to take the limit $\alpha \to 0$ in a second time, we need a quantitative rate of convergence.

We define the norm

$$\|g\|_0 := \sup_{(x,v)\in\mathbb{D}} \left|M^{-1}(v) g(x,v)\right| \text{ and } \|g\|_k := \sum_{|\alpha|\leq k} \|\nabla^\alpha g\|_0. \tag{2.14}$$

Theorem 2. *Let g and h be two test functions $\mathscr{C}^1(\mathbb{D})$, with $\|g\|_1, \|h\|_1 < \infty$. Then there exist three constants $C > 1$, $C' > 1$ and $\mathfrak{a} \in (0,1)$ independent of g, h such that for any ε small enough, $T > 1$, $\theta < \frac{1}{C'T^2}$, $\alpha, \mathfrak{d} \in (\log|\log\varepsilon|^{-1}, 1)$,*

$$\sup_{t\in[0,T]} \left|\mathbb{E}_\varepsilon\left[\zeta_\varepsilon^t(h)\zeta_\varepsilon^0(g)\right] - \int h(z)\mathbf{g}_\alpha(t,z)M(z)\,\mathrm{d}z\right| \leq C\left(C\frac{T^{3/2}\theta^{1/2}}{\mathfrak{d}^2} + \frac{T}{\theta}2^{T^2/\theta^2}\left(\frac{CT}{\mathfrak{d}}\right)^{2^{T/\theta}}\varepsilon^{\mathfrak{a}/2}\right)\|g\|_1\|h\|_1, \tag{2.15}$$

where $\mathbf{g}_\alpha(t,z)$ is the solution of the linearized Boltzmann equation

$$\begin{aligned} \partial_t \mathbf{g}_\alpha(t) + v\cdot\nabla_x \mathbf{g}_\alpha(t) &= \frac{1}{\mathfrak{d}}\mathscr{L}_\alpha \mathbf{g}_\alpha(t), \\ \mathbf{g}_\alpha(t=0) &= g \end{aligned} \tag{2.16}$$

The theorem is valid in any dimension $d \geq 2$. However, taking the grazing collision limit $\alpha \to 0$ has a physical meaning only in dimension 3.

This theorem is the main purpose of the present article. We conclude the proof of Theorem 2 by Estimation (3.26), and we outlined the main step of the proof in Section 2.7.

2.5. **Derivation of the linearized Landau equation and Boltzmann equation without cut-off.** The dimension is fixed at $d = 3$ (we think that it is the most physical case). In a second time, we want to take the weak coupling limit $\alpha \to 0$, $\mathfrak{d} \to 0$. It is the abstract of Raphael Winter and the author's result in [LBW22].

The case where the singularity $\frac{1}{r^s}$ of the potential is bigger than the Coulomb's case ($s > 1$). In the limit $\alpha \to 0$ we will only see the effects of the singularity at the origin. We define the power law potential $\mathscr{U}_s(r) := 1/r^s$. It is natural to guess that

$$\alpha^{-\frac{2}{s}}\mathscr{L}_\alpha \to \mathscr{L}_{\mathscr{U}_s} \tag{2.17}$$

which is a linearized Boltzmann operator associated without cutoff (see Appendix A for a rigorous definition of $\mathscr{L}_{\mathscr{U}_s}$ and a justification of the scaling $\mathfrak{d} = \alpha^{2/s}$).

Remark 2.2.3. *We say that the Boltzmann operator $\mathscr{L}_{\mathscr{U}_s}$ has no cutoff because particles can interact at long range and the collision kernel $b_s(v - v_*, \eta)$ associated to the potential $1/r^s$ (defined as in Remark 2.2.2) is not integrable in the η variable.*

The Coulomb case $s = 1$. It is not possible to define the Boltzmann operator for the Coulomb potential. However, we can prove (see [LBW22]) that for g a test function smooth enough,

$$\frac{1}{\alpha^2|\log\alpha|}\mathscr{L}_\alpha g \underset{\alpha\to 0}{\longrightarrow} c_{\mathscr{V}}\mathscr{K} g \tag{2.18}$$

where $c_{\mathscr{V}} = 1$ is a diffusion constant and $\mathscr{K}$ is the linearized Landau operator

$$\mathscr{K} g(v) = \frac{2\pi}{M(v)}\nabla_v \cdot \left(\int_{\mathbb{R}^3} \frac{P^\perp_{v-v_*}}{|v - v_*|}(\nabla g(v) - \nabla g(v_*))M(v)M(v_*)\,\mathrm{d}v_*\right). \tag{2.19}$$

Treat now the case lower than Coulomb $s \in [0,1)$. For these potentials, the scaling and the diffusion change:

$$\mathfrak{d} = \alpha^2, \quad 2\pi c_{\mathscr{V}} = \frac{1}{8\pi}\int_{\mathbb{R}^3} \delta(k\cdot\vec{e}_1)|k|^2|\hat{\mathscr{V}}(k)|^2\,\mathrm{d}k, \tag{2.20}$$

where $\vec{e}_1$ is a unit vector, and we use the convention $\hat{\mathscr{V}}(k) = \int_{\mathbb{R}^3} e^{-ik\cdot x}\mathscr{V}(x)\,\mathrm{d}x$ for the Fourier transform of $\mathscr{V}$, and

$$\frac{1}{\alpha^2}\mathscr{L}_\alpha g \underset{\alpha\to 0}{\longrightarrow} c_{\mathscr{V}}\mathscr{K} g \tag{2.21}$$

The previous discussion can be summarised by the following theorem:

Theorem 3 (proven in [LBW22]). *For $g : \mathbb{D} \to \mathbb{R}$ smooth and $\mathscr{V}$ respecting Assumption ??, there exists a positive constant C such that*

$$\left\|\mathfrak{d}_{s,\alpha}^{-1}\mathscr{L}_\alpha g - \mathscr{L}_\infty g\right\|_{L^2(M(v)\,\mathrm{d}z)} \leq \frac{C}{|\log\alpha|}\|g\|_3 \tag{2.22}$$

where $\mathscr{L}_\infty$ and $\mathfrak{d}_{s,\alpha}$ are given by

Singularity	*Mean free-path*	*Limiting operator*
$s>1$	$\mathfrak{d}_{s,\alpha} := \alpha^{2/s}$	$\mathscr{L}_\infty = \mathscr{L}_{\mathscr{U}_s}$
$s=1$	$\mathfrak{d}_{s,\alpha} := \alpha^2\lvert\log\alpha\rvert$	$\mathscr{L}_\infty = \mathscr{K}$
$s<1$	$\mathfrak{d}_{s,\alpha} := \alpha^2$	$\mathscr{L}_\infty = c_{\mathscr{V}}\mathscr{K}$, $c_{\mathscr{V}} = \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\delta(k\cdot\vec{e}_1)\lvert k\rvert^2\lvert\hat{\mathscr{V}}(k)\rvert^2\,\mathrm{d}k$

In addition, defining $\mathbf{g}_\infty(t)$ the solution of

$$\begin{aligned}\partial_t\mathbf{g}_\infty(t) + v\cdot\nabla_x\mathbf{g}_\infty(t) &= \mathscr{L}_\infty\mathbf{g}_\infty(t),\\ \mathbf{g}_\infty(t=0) &= g\end{aligned} \tag{2.23}$$

and $\mathbf{g}_\alpha$ the solution of (2.16) with $\mathfrak{d} := \mathfrak{d}_{s,\alpha}$, the following convergence holds

$$\mathbf{g}_\alpha \underset{\alpha\to 0}{\rightharpoonup}{}^{*}\ \mathbf{g}_\infty \text{ in } L^\infty_t(\mathbb{R}^+(L^2(M(v)\,\mathrm{d}z)). \tag{2.24}$$

Combining it with Theorem 2 we obtain the main theorem:

Theorem 4. *Let $f, g \in L^2(M(v)\,\mathrm{d}z)$ be two test functions.*

Consider a potential $\mathscr{V}$ such that the Assumptions 2.1 are verified and $\mathscr{V}(r) \underset{r\to 0^+}{\sim} \frac{1}{r^s}$, $s>1$.

Fix the scaling $\mu\varepsilon^2\mathfrak{d}_{s,\alpha} = 1$. Then we have the following convergence result: for all $t\geq 0$,

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] \underset{\substack{\varepsilon\to 0\\ \alpha\to 0\\ \alpha>\log|\log\varepsilon|^{-\frac{1}{4}}}}{\longrightarrow} \int h(z)\mathbf{g}_\infty(t,z)M(z)\,\mathrm{d}z$$

where $\mathbf{g}_\infty(t)$ is the solution of the equation (2.23).

Proof. First, the space $E := \{g : \mathbb{D}\to\mathbb{R},\ \|g\|_1 < \infty\}$ is dense in $L^2(M(v\,\mathrm{d}z))$.

Since the two bilinear operators

$$(h,g)\mapsto \mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right],\ (h,g)\mapsto\int h(z)\mathbf{g}_\infty(t,z)M(z)\,\mathrm{d}z$$

are continuous on $L^2(M(v)\,\mathrm{d}z)$ (see [BGSRS21]), it is sufficient to take $g,h\in E$.

Set $T := \max(1,t)$. Fixing $\theta := \frac{1}{\beta\log|\log\varepsilon|}$ for $\beta\in(0,1)$ small enough,

$$C\left(C\frac{T^{3/2}\theta^{1/2}}{\mathfrak{d}^2_{s,\alpha}} + \frac{T}{\theta}2^{T^2/\theta^2}\left(\frac{CT}{\mathfrak{d}_{\alpha,s}}\right)^{2^{T/\theta}}\varepsilon^{\mathfrak{a}/2}\right) = O(\varepsilon^{\mathfrak{a}/4}).$$

Hence Theorem 2 provides

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(g)\zeta^0_\varepsilon(h)\right] = \int h(z)\mathbf{g}_\alpha(t,z)M(v)\,\mathrm{d}z + O(\varepsilon^{\mathfrak{a}/4}\|g\|_1\|h\|_1).$$

Theorem 3 provides the convergence

$$\int h(z)\mathbf{g}_\alpha(t,z)M(v)\,\mathrm{d}z \to \int h(z)\mathbf{g}_\infty(t,z)M(v)\,\mathrm{d}z.$$

This conclude the proof. □

2.6. **Central Limit Theorem for the fluctuation field.** One can ask if it is possible to go further in the description of the fluctuation field. In the hard spheres setting, it is possible to prove a Central Limit Theorem for ζ_ε^t.

At time 0, one can show that ζ_ε^0 converges in law to the Gaussian field ζ^0 define by

Definition 2.2.2. *Let ζ^0 the Gaussian field on $\mathbb{D}$ of covariance*

$$\forall f, g \in L(M(v)dz), \quad \begin{cases} \mathbb{E}\left[\zeta^0(h)\zeta^0(g)\right] = \int h(z)g(z)M(v)dz, \\ \mathbb{E}\left[\zeta^0(g)\right] = 0. \end{cases} \tag{2.25}$$

It is possible to generalize this result to the time dependent process (see [Spo81, Spo83, BGSRS20] for short time result and [BGSRS22a] for long time result).

Theorem 5 (Bodineau, Gallagher, Saint-Raymond, Simonella, [BGSRS22a])**.** *Consider the hard spheres system in d-dimensional torus $\mathbb{T}^d$ (d bigger than 3) and fix the Boltzmann-Grad scaling $\mu\varepsilon^{d-1} = 1$. The fluctuation field $(\zeta_\varepsilon)_{t\geq 0}$ converges for al time to ζ^t, the Gaussian field solution of the fluctuating Boltzmann equation*

$$\begin{cases} d\zeta^t = \mathscr{L}_B\zeta^t dt + d\xi^t \\ \zeta^{t=0} = \zeta^0 \end{cases}. \tag{2.26}$$

The field ξ^t is the mean free Gaussian field of covariance

$$\begin{aligned} &\mathbb{E}\left[\int_0^T h(z_1)\xi^{\tau_1}(dz_1)d\tau_1 \int_0^T h(z_2)\xi^{\tau_2}(dz_2)d\tau_2\right] \\ &\qquad := \frac{1}{2}\int_0^T d\tau \int d\mu(z_1, z_2, \eta)M(v_1)M(v_2)\boldsymbol{\Delta}h\,\boldsymbol{\Delta}g, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} d\mu(z_1, z_2, \eta) &:= \delta_{x_1=x_2} b((v_1 - v_2), \eta)dz_1dz_2d\eta, \\ \boldsymbol{\Delta}h &:= h(v_1') + h(v_2') - h(v_1) - h(v_2), \end{aligned}$$

and $b((v_1 - v_2), \eta)$ the hard sphere collision kernel.

As the particle dynamics has a memory (it is a purely deterministic process), it does not preserve the initial Gaussian structure. In [BGSRS22a], the Gaussian property is proved by checking asymptotically the Wick's law for the limiting field.

One can asks if such a theorem still applied in a system of particle interacting through a more general potential $\mathscr{V}$. For the moment the question is still open.c

Notations. For $m < n$ two integers, we denote $[m, n] := \{m, m+1, \cdots, n\}$ and $[n] := [1, n]$.

For $Z_n \in \mathbb{D}^n$, and $\omega \subset [n]$, we denote

$$Z_\omega := (z_{\omega(1)}, \cdots, z_{\omega(|\omega|)})$$

where $\omega(i)$ is the i-th element of ω counted in increasing order.

Given a family particles indices $\{i_1, \cdots, i_n\}$, the notation $(i_1, \cdots, i_n)$ indicates the ordered sequence in which $\forall k \neq l$, $i_k \neq i_l$. In addition

- $\underline{i}_n := (i_1, \cdots, i_n)$,
- for $m \leq n$, $\underline{i}_m = (i_1, \cdots, i_m)$, and more generally for $\omega \subset [1, n]$, $\underline{i}_\omega := (i_{\min\omega}, \cdots, i_{\max\omega})$,
- for $0 \leq m < n$ and $(i_1, \cdots, i_m)$, $\sum\limits_{(i_{m+1}, \cdots, i_n)}$ denotes the sum over every family $(i_{m+1}, \cdots, i_n)$ such that for $1 \leq k < l \leq n$, $i_k \neq i_l$, and
$$\sum_{\underline{i}_n} = \sum_{(i_1, \cdots, i_n)},$$
- $\mathbf{Z}_{\underline{i}_n} := (\mathbf{z}_{i_1}, \cdots, \mathbf{z}_{i_n})$, as ordered sequence.

We also precise the sense of Landau[1] notation: $A = B + O(D)$ means that there exists a constant C depending only on the dimension such that $|A - B| < C\,D$.

[1]from Edmund Landau and not Lev Landau.

When we performed estimation, C is a positive constant (which can change from a line to another) and the final time t is supposed to be bigger than 1 (in general, we prefer to denote τ any intermediate time).

Finally, let h_n be a function on $\mathbb{D}^n$. We denote

$$\mathbb{E}_\varepsilon\left[h_n\right] := \mathbb{E}_\varepsilon\left[\frac{1}{\mu^n}\sum_{(i_1,\cdots,i_n)} h_n\left(\mathbf{Z}_{\underline{i}}\right)\right]$$

and the associated centered function defined on $\mathscr{D}_\varepsilon$

$$\hat{h}_n(\mathbf{Z}_{\mathscr{N}}) := \frac{1}{\mu^n}\sum_{(i_1,\cdots,i_n)} h_n\left(\mathbf{Z}_{\underline{i}_n}\right) - \mathbb{E}_\varepsilon\left[h_n\right].$$

2.7. **Strategy of the proof Theorem 2.** The proof of Theorem 2 follows a path similar to [BGSRS21] and [LB22], which were written in the case of hard spheres.

As $\zeta_\varepsilon^0(g)$ is a mean-free random variable on $\mathscr{D}$, we can write

$$\mathbb{E}_\varepsilon\left[\zeta_\varepsilon^t(h)\zeta_\varepsilon^0(g)\right] = \frac{1}{\sqrt{\mu}}\mathbb{E}_\varepsilon\left[\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t))\zeta_\varepsilon^0(g)\right]. \tag{2.28}$$

We see that the function h is evaluated at time t whereas the function g is evaluated at time 0. The first step of the proof is the construction of a family of functionals $(\Phi_{1,n})_{1\leq n}$, $\Phi_{1,n}^t : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D}^n)$ such that for any initial configuration $\mathbf{Z}_{\mathscr{N}} \in \mathscr{D}$,

$$\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t)) = \sum_{n\geq 1}\sum_{\underline{i}_n} \Phi_{1,n}^t[h](\mathbf{Z}_{\underline{i}_n}(0)). \tag{2.29}$$

The first part of Section 3 is dedicated to give an explicit expression to the $\Phi_{1,n}^t$:

$$\Phi_{1,n}^t[h](Z_n) := \frac{1}{(n-1)!}\sum_{\text{history}} h(z_1(t, Z_n, \text{history}))\mathbb{1}_{\text{history}}\sigma(\text{history}) \tag{2.30}$$

where $z_1(t, Z_n, \text{history})$ is the final position of particle 1 in the *pseudorajectory* of prescribed *history*. A pseudotrajectory is the path of a finite set of particles. This set is divided into disjoint clusters. When a particle meets another one, they interact if they are in the same cluster and ignore each other else. In the both cases, this meeting creates a link between the two trajectories. The history of the pseudocharacteristic is a discrete parameter describing the links between the particles[2]. In the preceding formula $\mathbb{1}_{\text{history}}$ checks that the pseudotrajectory is possible, and $\sigma(\text{history}) = \pm 1$ is a sign link to a splitting of the collision operator $\mathscr{L}_\alpha$ into a positive and a negative part.

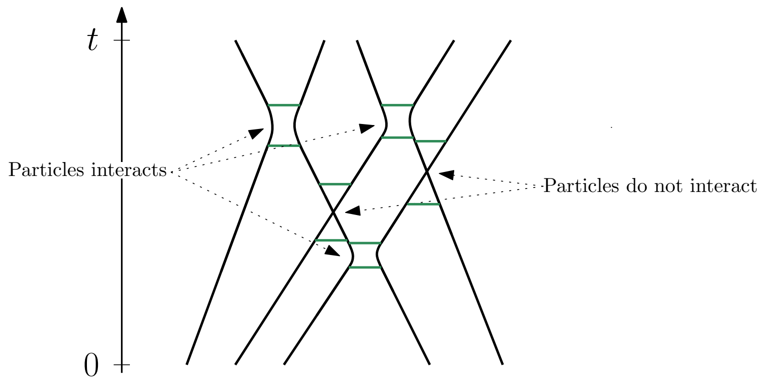

FIGURE 3. Exemple of pseudotrajectory for four particles. Note that there are two clusters of particles and $\sigma(\text{history}) = 1$.

The decomposition (2.29) can be interpreted as the dual formulation of the *pseudocharacteristic development* first used by Lanford in his original proof [Lan75] for the hard sphere system and later adapted by

[2]An history can be constructed as first a partition into clusters $(\rho_1, \cdots, \rho_r)$ of the set of particles and of the graph of vertices $[r]$ and with the edge (i,j) if one particle of ρ_i meets one particle of ρ_j. The sign $\sigma(\text{history})$ is 1 if the number of edges is even, -1 else.

King for other potentials [Kin75]. However, the King decomposition makes a distinction between binary interaction and interaction with more than two particles. In order to avoid this feature, we prefer to use the dynamical cluster introduced by Sinai [Sin72] in a different setting and later by Bodineau *et al.* for hard spheres in Boltzmann-Grad scaling (see [BGSRS22b]).

In the classical derivation of the Boltzmann equation (and here of the linearized Boltzmann equation), there are two main steps. First, we need to prove that each term

$$\mathbb{E}_\varepsilon\left[\mu^{-\frac{1}{2}}\sum_{\underline{i}_n}\Phi^t_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(0))\zeta^0_\varepsilon(g)\right] \tag{2.31}$$

converges to its formal limit. It is defined by the asymptotic of the Hamiltonian dynamics when particles become punctual.

The main obstacles to this convergence are *multiple interactions* (interaction between more than three particles) and *recollisions*. A recollision is an interactions between two particles q and $\bar{q}$, beginning at time τ and such that we can find a sequence a sequence of particles $q_1 = q, q_2, \cdots, q_r = \bar{q}$ with q_i meeting q_{i+1} before time τ. Recollisions become rare in the limit $\varepsilon \to 0$ and are impossible in the limiting process.

The second step is an *a priori* bound of the terms of the series (2.29). An L^1 estimation is used in the classical derivation of the Boltzmann equation (see [Lan75, Kin75, GSRT13, PSS14]). It is valid only for short times. The linear version of the problem (one tagged particle followed in a background initially at equilibrium) is only a $O(1)$ perturbation of equilibrium. Thus, the L^1 bounds are valid for all time (see [vBLLS80, BGSR16, Ayi17, Cat18]). The linearized setting is a $O(\mu)$ perturbation of the equilibrium and L^1 bounds are no longer sufficient to reach long time (Spohn used them to describe the fluctuation on short time in [Spo81]). To gain estimation on a longer time interval, it is convenient to consider L^2 estimates (see [BGSR17, BGSRS21, BGSRS22a, LB22]). Indeed, because $\zeta^0_\varepsilon(g)$ is a mean free random variable, for any intermediate time $t_s \in [0,t]$,

$$\begin{aligned}\left|\mathbb{E}_\varepsilon\left[\mu^{-\frac{1}{2}}\sum_{\underline{i}_n}\Phi^{t-t_s}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\right]\right| &= \left|\mathbb{E}_\varepsilon\left[\mu^{n-\frac{1}{2}}\hat{\Phi}^{t-t_s}_{1,n}[h](\mathbf{Z}_{\mathscr{N}}(t_s))\zeta^0_\varepsilon(g)\right]\right| \\ &\leq \mathbb{E}_\varepsilon\left[\mu^{2n-1}\left(\hat{\Phi}^{t-t_s}_{1,n}[h](\mathbf{Z}_{\mathscr{N}}(t_s))\right)^2\right]^{\frac{1}{2}}\mathbb{E}_\varepsilon\left[\left(\zeta^0_\varepsilon(g)\right)^2\right]^{\frac{1}{2}} \\ &\leq \mathbb{E}_\varepsilon\left[\mu^{2n-1}\left(\hat{\Phi}^{t-t_s}_{1,n}[h](\mathbf{Z}_{\mathscr{N}}(0))\right)^2\right]^{\frac{1}{2}}\mathbb{E}_\varepsilon\left[\left(\zeta^0_\varepsilon(g)\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

using a Cauchy-Schwartz inequality and the invariance of the Gibbs measure. Hence, it is possible to begin a development along pseudotrajectories and stop at time t_s when it becomes "pathological". Then we can ignore what happens in the time interval $[0, t_s]$.

We need to bound the $\mathbb{E}_\varepsilon\big[\big(\hat{\Phi}^{t-t_s}_{1,n}[h]\big)^2\big]$, which is linked to the estimation of the integrals (see the Section 4)

$$\forall m \leq n \int \left|\Phi^{t-t_s}_{1,n}[h](Z_n)\Phi^{t-t_s}_{1,n}[h](Z_{[n-m,2n-m]})\right| e^{-\mathscr{H}_{2n-m}(Z_{2n-m})}\,\mathrm{d}Z_{2n-m}. \tag{2.32}$$

Unfortunately, we do not know how to take account of the signs σ(history) in the bound of $\Phi^{t-t_s}_{1,n}$. Thus, we are reduced to counting the number of *histories* needed to describe the pseudotrajectories with n particles. A useful tool is the collision graph on $[\tau,\tau']$: its vertices are the particles $1,\cdots,n$ and it has an edge $(q,\bar{q})$ for any collision involving particles q and $\bar{q}$ happening between times τ and τ'. If we forbid multiple interactions and recollisions, the collision graph on $[t_s,t]$ has no cycle. Hence, we only need $n-1$ parameters to decide for each collision if particles interact or not.

We now introduce two samplings, one to control regular collisions and one to control recollisions and multiple interactions.

The first sampling has a relatively large step $\theta := \frac{1}{C\log|\log\varepsilon|}$ (for some constant C large enough). We stop the pseudotrajectories development at time $t-k\theta$ if there are more than 2^k particles involved in the pseudotrajectory. Hence, the number of particles at time 0 remains controlled.

The second sampling has a shorter step, $\delta := \varepsilon^{1/12}$. We stop the expansion at time $t_s := t-k\delta$ if the pseudotrajectory has at least one recollision on $[t_s,t]$ (but no recollision on $[t_s+\delta,t]$). Imposing recollisions create an additional geometric condition, and thus, an extra-smallness gain.

However, we still have too much possible history. To reduce their number, we separate the pseudotrajectories in two categories. In *non-pathological* pseudotrajectories, the collision graphs on $[t_s,t_s+\delta]$ and on $[t_s+\delta, t]$ have both no cycle. We are in a setting close to the case without recollision, and we only need C^n parameters (C a fixed constant) to describe the histories.

We explain now how to treat the pathological recollisions part. We condition the initial data $\mathbf{Z}_{\mathscr{N}}(0)$ such that on each interval $[k\delta,(k+1)\delta]$ a particle can *interact*[3] with only a finite number of particles γ (we will take $\gamma := 12d$). Hence, for a pseudotrajectory $\mathrm{z}_1(t, Z_n, \text{history})$, the history has to describe first a partition of $[n]$ into small clusters of particles that interact together on $[0,\delta]$ and how they really interact. As the size of each cluster is uniformly bounded, the number of histories is at most of order C^n for some $C > 1$.

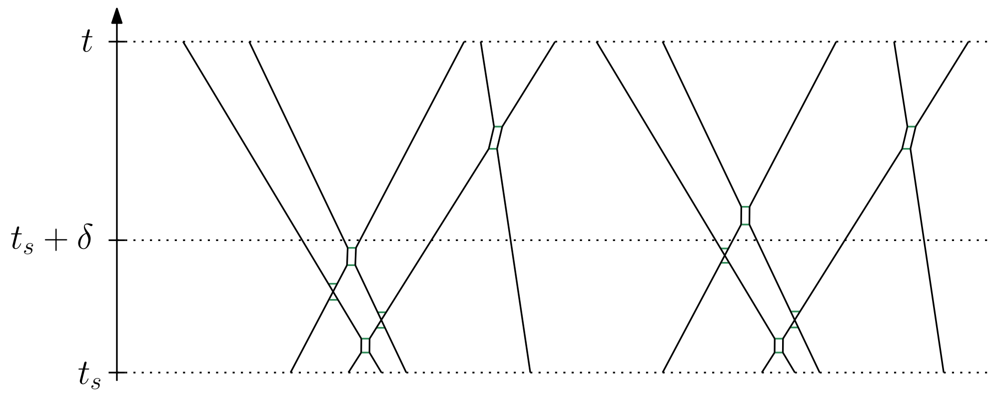

FIGURE 4. An example of one pathological pseudotrajectory (on the left) and a non-pathological one (on the right)

The paper is organised as follows: In section 3, we give a proper definition of histories and we use it to construct the functionals $\Phi^t_{1,n}$. Then we implement the two samplings. This allows to decompose $\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right]$ into a main term, plus error terms of different nature: a development on pseudotrajectories (i) without recollisions (bounded in Section 5), (ii) with non-pathological recollisions (bounded in section 7) and (iii) with pathological recollisions (bounded in Section 8). The estimation of the error terms requires standard $L^2(\mathbb{P}_\varepsilon)$ estimates based on static cumulant decompositions, which are reported in Section 4. Finally, the convergence of the main term is proven in Section 6. Annex B to the analyses of trajectories leading to recollisions of multiple interactions.

3. Development along pseudotrajectories and time sampling

3.1. **Dynamical cluster development.** For any test functions h and $g : \mathbb{D} \to \mathbb{R}$ we want to compute

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] = \frac{1}{\mu}\mathbb{E}_\varepsilon\left[\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t))\sum_{j=1}^{\mathscr{N}} g(\mathbf{z}_j(0))\right].$$

We have a sum evaluated at time t and a sum evaluated at time 0. In order to compute it, we have to pull back the second sum to time 0: we want to construct a family of applications $\Phi^t_{1,n} : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D}^n)$ such that for almost all initial data $\mathbf{Z}_{\mathscr{N}}(0) \in \mathscr{D}$

$$h(\mathbf{z}_{i_1}(t)) = \sum_{n\geq 1}\sum_{(i_2,\cdots,i_n)} \Phi^t_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(0)).$$

More generally, we will construct a family of functional $\Phi^t_{m,n} : L^\infty(\mathbb{D}^m) \to L^\infty(\mathbb{D}^n)$ (with $m < n$) such that for any test functions $h_m \in L^\infty(\mathbb{D}^m)$,

$$h_m(\mathbf{Z}_{\underline{i}_m}(t)) = \sum_{n\geq 1}\sum_{(i_{m+1},\cdots,i_n)} \Phi^t_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)).$$

Remark 3.0.1 (Comparison with the hard sphere setting)**.** *In the hard spheres setting, a tree pseudotrajectories development is used as it comes directly from the BBGKY hierarchy (see for example* [Lan75, PS15, BGSRS21, LB22]*). We begin at time* 0 *with n particles, and at each collision, we can remove or not one particle to end at time t with m particles. However, in the case of physical potential, writing the BBGKY hierarchy is difficult as particles can overlap, and there can be interaction between*

[3] the meaning of *interact* will be precise in definition 3.0.5.

more than three particles are possible (see [Kin75, GSRT13, PSS14] *for a description of the BBGKY hierachy). Hence, we will use a different kind of pseudotrajectory development called "dynamical cluster development" (see* [Sin72, PSW16, BGSRS22b, Gra58], *from which we take inspiration).*

Fix $\lambda \subset \mathbb{N}$ a finite set of particles. We denote by $\mathsf{Z}^\lambda(\tau) = (\mathsf{X}^\lambda(\tau), \mathsf{V}^\lambda(\tau))$ the Hamiltonian trajectory, linked to the energy

$$\mathscr{H}_\lambda(Z_\lambda) := \sum_{q\in\lambda} \frac{|v_q|^2}{2} + \frac{\alpha}{2} \sum_{\substack{q,\bar q\in\lambda\\ q\neq \bar q}} \mathscr{V}\left(\frac{x_q - x_{\bar q}}{\varepsilon}\right)$$

of the particles λ (isolated of the other particles) with initial data $\mathsf{Z}^\lambda(0) = Z_\lambda$. For any subset $\lambda' \subset \lambda$, we denote $\mathsf{Z}^\lambda_{\lambda'}(\tau)$ is the trajectory of particles λ' in $\mathsf{Z}^\lambda(\tau)$.

Definition 3.0.1. *Given $Z_\lambda \in \mathbb{D}^{|\lambda|}$, we construct the graph G with vertex λ and $(q,\bar q) \in \lambda^2$ is an edge if and only if $q < \bar q$ and if there exists a time $\tau \in [0,t]$ such that*

$$\exists \tau \in [0,t],\ \left|\mathsf{x}^\lambda_q(\tau) - \mathsf{x}^\lambda_{\bar q}(\tau)\right| \leq \varepsilon$$

We say that $\mathsf{Z}^\lambda(\tau)$ form a dynamical cluster *if the graph G is connected. We denote $\Delta\!\!\!\Delta_{|\lambda|}(Z_\lambda)$ the indicator function that the trajectory $\mathsf{Z}^\lambda(\tau)$ forms a dynamical cluster.*

*In the same way, for $\omega \subset \lambda$, we say that $\mathsf{Z}^\lambda(\tau)$ form a ω-*cluster *if, in the collision of $\mathsf{Z}^\lambda(\tau)$, all the particles are in the same connected components of G that one of the particles of ω. The function $\Delta\!\!\!\Delta^\omega_{|\lambda|}(Z_\lambda)$ is equal to 1 if $\mathsf{Z}^\lambda(\tau)$ is a ω-cluster, 0 else.*

Remark 3.0.2. *In the following, we consider that all the graphs are unoriented.*

Definition 3.0.2. *We say that trajectories $\mathsf{Z}^\lambda(\tau)$ and $\mathsf{Z}^{\lambda'}(\tau)$ (with $\lambda \cap \lambda' = \emptyset$) have an overlap if there exist a couple of particle $(q,q') \in \lambda\times\lambda'$ and some time $\tau\in[0,t]$, such that $|\mathsf{x}^\lambda_q(\tau) - \mathsf{x}^{\lambda'}_{q'}(\tau)| \leq \varepsilon$. Then we denote $\lambda \overset{\circ}{\sim} \lambda'$.*

For $(Z_{\lambda_1}, \cdots, Z_{\lambda_{\mathbf{l}}}) \in \prod_{i=1}^{\mathbf{l}} \mathbb{D}^{|\lambda_i|}$ initial data, we look at the indicator function that for any $i \neq j$, $\mathsf{Z}^{\lambda_i}(\tau)$ and $\mathsf{Z}^{\lambda_j}(\tau)$ have no overlap. We can expand it as

$$\prod_{1\leq i<j\leq \mathbf{l}} \left(1 - \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}\right) = \sum_{\substack{\omega\subset[1,l]\\ 1\in\omega}} \underbrace{\sum_{C\in\mathscr{C}(\omega)} \prod_{(i,j)\in E(C)} -\mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}}_{:=\mathbb{O}_{|\omega|}(Z_{\lambda_1}, Z_{\lambda_{\omega(2)}}, \cdots, Z_{\lambda_{\omega(|\omega|)}})} \prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}} \left(1 - \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}\right). \tag{3.1}$$

We have defined $(\mathbb{O}_l)_l$ the cumulants of the overlap indicator functions.

We make a partition of $\mathscr{D}$ depending on the way particles interact during the time interval $[0,t]$: fixing $\mathscr{N} \in \mathbb{N}$ and $\underline{i}_m$,

$$\begin{aligned} h_m(\mathbf{Z}_{\underline{i}_m}(t)) &= \sum_{\mathbf{l}=1}^{\mathscr{N}} \sum_{\substack{\underline{i}_m\subset\lambda_1\\ (\lambda_2,\cdots,\lambda_{\mathbf{l}})\in\mathscr{P}^{\mathbf{l}-1}_{\lambda_1^c}}} h_m(\mathbf{Z}_{\underline{i}_m}(t)) \Delta\!\!\!\Delta^{\underline{i}_m}_{\lambda_1}(\mathbf{Z}_{\lambda_1}) \prod_{i=2}^{\mathbf{l}} \Delta\!\!\!\Delta_{|\lambda_i|}(\mathbf{Z}_{\lambda_i}) \prod_{1\leq i<j\leq\mathbf{l}} \left(1 - \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}\right) \\ &= \sum_{\mathbf{l}=1}^{\mathscr{N}} \sum_{\substack{\underline{i}_m\subset\lambda_1\\ (\lambda_2,\cdots,\lambda_{\mathbf{l}})\in\mathscr{P}^{\mathbf{l}-1}_{\lambda_1^c}}} h_m(\mathbf{Z}_{\underline{i}_m}(t)) \Delta\!\!\!\Delta^{\underline{i}_m}_{\lambda_1}(\mathbf{Z}_{\lambda_1}) \prod_{i=2}^{\mathbf{l}} \Delta\!\!\!\Delta_{|\lambda_i|}(\mathbf{Z}_{\lambda_i}) \sum_{\substack{\omega\subset[1,l]\\ 1\in\omega}} \mathbb{O}_{|\omega|}(\mathbf{Z}_{\underline{\lambda}_\omega}) \\ &\qquad\qquad \times \prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}} \left(1 - \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}\right) \end{aligned}$$

where we have denoted $\mathscr{P}^r_\omega$ the set of the unordered partitions $(\rho_1, \cdots, \rho_r)$ of the set ω.

We make the change of variables

$$(\mathbf{l}, (\lambda_1, \cdots, \lambda_{\mathbf{l}}), \omega) \mapsto \left(\rho, \mathbf{l}_1, (\bar\lambda_1, \cdots, \bar\lambda_{\mathbf{l}_1}), \mathbf{l}_2, \left(\tilde\lambda_1, \cdots, \tilde\lambda_{\mathbf{l}_2}\right)\right)$$

where

$$\rho := \bigcup_{i\in\omega} \lambda_i,\ \mathbf{l}_2 := |\omega|,\ \mathbf{l}_1 := \mathbf{l} - |\omega|,\ (\bar\lambda_1, \cdots, \bar\lambda_{\mathbf{l}_1}) := (\lambda_j)_{j\in\omega^c} \text{ and } \left(\tilde\lambda_1, \cdots, \tilde\lambda_{\mathbf{l}_2}\right) := (\lambda_j)_{j\in\omega}.$$

The set ρ is the set of particles linked to $\underline{i}_m$ *via* a chain of interactions or overlaps. We get

$$\begin{aligned} &h_m(\mathbf{Z}_{\underline{i}_m}(t)) \\ &\quad= \sum_{\underline{i}_m\subset\rho}\sum_{\mathfrak{l}_1=1}^{|\rho|}\sum_{\substack{\underline{i}_m\subset\bar\lambda_1\subset\rho\\(\bar\lambda_2,\cdots,\bar\lambda_{\mathfrak{l}_1})\in\mathscr{P}^{\mathfrak{l}_1-1}_{\bar\lambda_1^c}}} h_m\left(\mathsf{Z}^{\bar\lambda_1}_{\underline{i}_m}(t)\right)\Delta\!\!\!\!\Delta^{\underline{i}_m}_{\bar\lambda_1}\left(\mathbf{Z}_{\bar\lambda_1}\right)\prod_{i=2}^{\mathfrak{l}_1}\Delta\!\!\!\!\Delta_{|\bar\lambda_i|}\left(\mathbf{Z}_{\bar\lambda_i}\right)\varphi_{\mathfrak{l}_1}\left(\mathbf{Z}_{\bar\lambda_1},\cdots,\mathbf{Z}_{\bar\lambda_{\mathfrak{l}_1}}\right) \\ &\qquad\times\sum_{\mathfrak{l}_2=1}^{|\rho^c|}\sum_{(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathfrak{l}_2})\in\mathscr{P}^{\mathfrak{l}_2}_{\rho^c}}\prod_{i=1}^{\mathfrak{l}_2}\Delta\!\!\!\!\Delta_{|\tilde\lambda_i|}(\mathbf{Z}_{\tilde\lambda_i})\prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}}\left(1-\mathbb{1}_{\tilde\lambda_i\overset{\circ}{\sim}\tilde\lambda_j}\right). \end{aligned}$$

The second line is the sum over all possible partitions $(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathfrak{l}_2})$ of ρ^c of the indicator function that they are effectively the dynamical cluster of the initial data. Hence, it is equal to one. We identify the n-th *dynamical cumulant* as

$$\begin{aligned}\Phi^t_{m,n}[h_m](Z_n) := \frac{1}{(n-m)!}\sum_{\mathfrak{l}=1}^{n}\sum_{[m]\subset\lambda_1\subset[n]}\sum_{\substack{(\lambda_2,\cdots,\lambda_{\mathfrak{l}})\\ \in\mathscr{P}^{\mathfrak{l}-1}_{\lambda_1^c}}} & h_m(\mathsf{Z}^{\lambda_1}_{[m]}(t))\varphi_{l}(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathfrak{l}}}) \\ &\times\Delta\!\!\!\!\Delta^{[m]}_{|\lambda_1|}(Z_{\lambda_1})\prod_{i=2}^{\mathfrak{l}}\Delta\!\!\!\!\Delta_{|\lambda_i|}(Z_{\lambda_i}),\end{aligned}\tag{3.2}$$

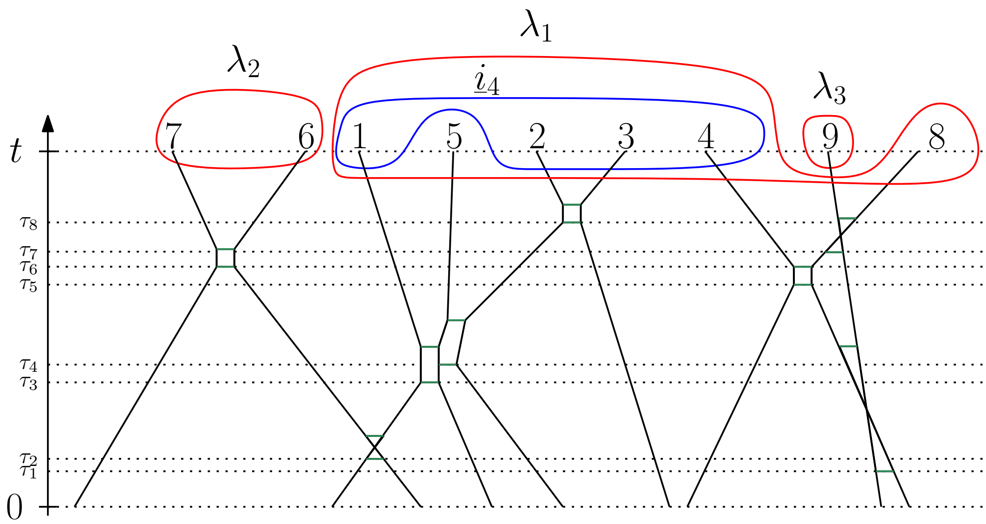

FIGURE 5. Example of trajectory in a dynamical cumulant. We want to follow the particles $\{1,2,3,4\}$.

We can now write the dynamical cluster expansion:

Theorem 6. *For almost all $\mathbf{Z}_{\mathscr{N}}\in\mathscr{D}$ we have*

$$h_m\left(\mathbf{Z}_{\underline{i}_m}(t)\right) = \sum_{n\geq m}\sum_{(i_{m+1},\cdots,i_n)}\Phi^t_{m,n}[h_m]\left(\mathbf{Z}_{\underline{i}_n}(0)\right). \tag{3.3}$$

Definition 3.0.3 (First type of pseudotrajectory)**.** *In the following, for a given $m\leq n$, $\underline{\lambda}=(\lambda_1,\cdots,\lambda_{\mathfrak{l}})$ a partition of $[n]$, we denote $\mathsf{Z}(t,Z_n,\underline{\lambda})$ the trajectory of the n particles following the Hamiltonian dynamics linked to*

$$\mathscr{H}_{\underline{\lambda}}(Z_n) := \sum_{\ell=1}^{\mathfrak{l}}\mathscr{H}_{\lambda_\ell}(Z_{\lambda_\ell}).$$

We define now the notion of collision graph:

Definition 3.0.4. *Fix $m\leq n$, collision parameters $\underline{\lambda}:=(\lambda_1,\cdots,\lambda_{\mathfrak{l}})$ and an initial position $Z_n\in\mathbb{D}^n$.*

We construct the collision graph *with vertex $[n]$ and with labeled edges of the form $(i,j)_{\tau,s}$, $\tau\in[0,t]$, $s\in\{\pm1\}$. The edges $(i,j)_{\tau,s}$ is in the graph if*

- $\tau\in(0,t)$, $|\mathsf{x}_i(\tau,\underline{\lambda})-\mathsf{x}_j(\tau,\underline{\lambda})|=\varepsilon$, $(\mathsf{x}_i(\tau,\underline{\lambda})-\mathsf{x}_j(\tau,\underline{\lambda}))\cdot(\mathsf{v}_i(\tau,\underline{\lambda})-\mathsf{v}_j(\tau,\underline{\lambda}))>0$,

- *or* $\tau = 0$, $\left|\mathsf{x}_i(0,\bar\lambda) - \mathsf{x}_j(0,\bar\lambda)\right| < \varepsilon$,
- $s = 1$ *if* i *and* j *are in the same* λ_k, $s = -1$ *else.*

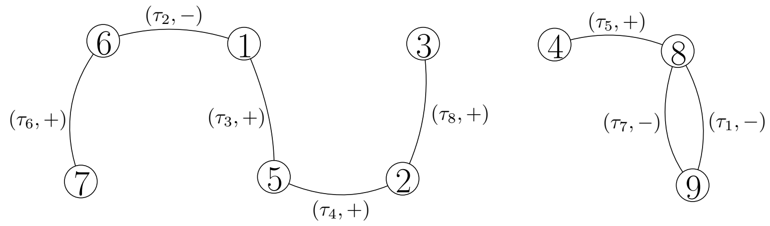

FIGURE 6. The collision graph link to the pseudotrajectory of Figure 3.1.

Remark 3.0.3. *Fix* $(\lambda_1, \cdots, \lambda_\ell)$ *a partition of* $[n]$*. Using Penrose's tree inequality (see* [Pen63, BGSRS20, Jan]*) the cumulant function* $\varpi_n(Z_{\lambda_1}, \cdots, Z_{\lambda_n})$ *is bounded by*

$$|\varpi_n(Z_{\lambda_1}, \cdots, Z_{\lambda_n})| \leq \sum_{T \in \mathscr{T}([\ell])} \prod_{(i,j)\in T} \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j} \tag{3.4}$$

where $\mathscr{T}([\ell])$ *is the set of simply connected graph on* $[\ell]$*. The case equality is reached, so we cannot expect a good* L^∞ *bound of* ϖ_n*.*

We introduce another parameterization of the pseudotrajectories to avoid this difficulty.

3.2. **Conditioning.** We describe now the conditioning used to control the pathological recollisions describe in the Section 2.7.

Definition 3.0.5 (Possible cluster)**.** *Let* $Z_r \in \mathbb{D}^r$ *an initial configuration. Consider* $\omega_1, \cdots, \omega_p$ *a family of subsets of* $[r]$ *such that*

$$\bigcup_{i=1}^{p} \omega_i = [r],$$

and $(\underline{\lambda}_i)_{i\leq p} = (\lambda_i^1, \cdots, \lambda_i^{l_i})_{i\leq p}$ *a family of partitions of* ω_i*. We denote* $\mathscr{G}_i$ *the collision graph of the pseudotrajectory* $\mathsf{Z}(\tau, Z_{\omega_i}, \underline{\lambda}_i)$ *on the time interval* $[0,\delta]$*. The graph* $\mathscr{G}$ *is the merge of all the* $\mathscr{G}_i$*.*

We say that Z_r *form a* possible cluster *if there exist some* $(\omega_i)_i$*,* $(\underline{\lambda}_i)_i$ *such that the graph* $\mathscr{G}$ *is connected.*

Let $\gamma > 0$ be an integer depending only on the dimension, $\delta > 0$ a time scale, and $\mathbb{V} > 0$ a velocity bound. We construct $\Upsilon_\varepsilon \subset \mathscr{D}$ the set of particle configurations such that for any time $\tau \in \{0, \delta, 2\delta, \cdots, t\}$, there is no possible cluster of size bigger than γ at time τ, and inside any subset of particles $\omega \subset [1, \mathscr{N}]$ whith less than γ elements, $\frac{1}{2}\|\mathbf{V}_\omega(\tau)\|^2$ is bounded by $\frac{1}{2}\mathbb{V}^2$. We have the following bound on the measure of the complement of Υ_ε:

Proposition 3.1. *There exists a constant* C_γ *depending only on* γ *and on the dimension such that*

$$\mathbb{P}_\varepsilon\left(\Upsilon_\varepsilon^c\right) \leq C_\gamma \frac{t}{\delta}\left(\mu\delta^\gamma + \mu^\gamma e^{-\mathbb{V}^2/4}\right). \tag{3.5}$$

Proof. We take the notation of the definition 3.0.5. If Z_r is a possible cluster, we consider $(\omega_i)_i$, $(\underline{\lambda}_i)_i$ such that the graph $\mathscr{G}$ is connected. We consider the first collision $(q,\bar q)_{\tau,s}$ such that for any $\tau' > \tau$, the graph $\mathscr{G}$ without the collisions after τ' is still connected. Hence, the graph $\mathscr{G}$ restricted to collisions happening during $[0,\tau[$ has two connected components, ϖ_1 and ϖ_2, and Z_{ϖ_1} and Z_{ϖ_2} are both possible clusters. We conclude that it is possible to find a $\varpi \subset [r]$ with $|\varpi| \geq \lceil \frac{r}{2} \rceil$ such that Z_r is a possible cluster.

$$\begin{aligned}
\mathbb{P}_\varepsilon\left(\Upsilon_\varepsilon^c\right) &\leq \sum_{k=0}^{t/\delta} \mathbb{E}_\varepsilon\left[\sum_{n=\gamma+1}^{2(\gamma+1)} \frac{1}{n!} \sum_{(i_1,\cdots,i_n)} \mathbb{1}_{\substack{\mathbf{Z}_{\underline{i}_n}(k\delta)\text{ form a}\\ \text{distance cluster}}} + \sum_{n=1}^{\gamma} \frac{1}{n!} \sum_{(i_1,\cdots,i_n)} \mathbb{1}_{\|\mathbf{V}_{\underline{i}_n}(k\delta)\|\geq\mathbb{V}}\right] \\
&\leq \frac{t}{\delta}\left(\sum_{n=\gamma+1}^{2(\gamma+1)} \frac{1}{n!}\mu^n \int \mathbb{1}_{\substack{Z_n\text{ form a}\\ \text{distance cluster}}} M^{\otimes n}\, \mathrm{d}Z_n + \sum_{n=1}^{\gamma} \frac{1}{n!}\mu^n \int \mathbb{1}_{\|V_{k'}\|\geq\mathbb{V}} M^{\otimes n}\, \mathrm{d}Z_n\right).
\end{aligned}$$

Using that (see Lemma B.3)

$$\int \mathbb{1}_{\substack{Z_n \text{ form a} \\ \text{possible cluster}}} M^{\otimes n} \,\mathrm{d}Z_n \leq C_\gamma \mu^{-n+1} \delta^{n-1}, \tag{3.6}$$

$$\int \mathbb{1}_{\|V_{k'}\| \geq \mathbb{V}} M^{\otimes n} \,\mathrm{d}Z_n \leq C_n e^{-\frac{\mathbb{V}^2}{4}} \tag{3.7}$$

we obtain the expected result.

We used that the Gibbs measure is time invariant. □

Hence, if we fix $\delta := \varepsilon^{1/12}$, $\mathbb{V} := |\log \varepsilon|$ and fix $\gamma = 24d$, $\mathbb{P}_\varepsilon(\Upsilon_\varepsilon^c)$ is $O(\varepsilon^d)$.

3.3. **The main part of the cumulant.** We define three kinds of pathology for the pseudotrajectories.

Definition 3.1.1. *Fix $m \leq n$, collision parameters $(\lambda_1, \cdots, \lambda_\mathfrak{l})$ and an initial position $Z_n \in \mathbb{D}^n$.*

- *There is an overlap if there are two particles q, q' and a time $\tau \in \delta\mathbb{Z} \cap [0,t]$ such that $|\mathsf{x}_q(\tau) - \mathsf{x}_{q'}(\tau)| \leq \varepsilon$.*
- *Fix a time τ and particles $i_i, \cdots i_k$. We define a graph G^τ with vertex $\{i_1, \cdots i_k\}$, and where (i_a, i_b) is an edge if and only if*

$$\left|\mathsf{x}_{i_a}(\tau, \underline{\lambda}) - \mathsf{x}_{i_b}(\tau, \underline{\lambda})\right| \leq \varepsilon.$$

  There is a multiple interaction *between $i_i, \cdots i_k$ at time τ if G^τ is connected.*
- *Fix $Z_n \in \mathbb{D}^n$ such that there is no multiple interaction during $[0,t]$.*

  We say that there is a recollision *if the collision graph has a cycle.*

These pathological cases are negligible in the limit $\varepsilon \to 0$. Hence, we can consider them as an error term.

In the following we define $\Phi^{0,t}_{m,n}$ as the part of $\Phi^t_{m,n}$ with only non pathological pseudotrajectories

$$\Phi^{0,t}_{m,n}[h_m](Z_n) := \frac{1}{(n-m)!} \sum_{\mathfrak{l}=1}^{n} \sum_{[m] \subset \lambda_1 \subset [n]} \sum_{\substack{(\lambda_2, \cdots, \lambda_\mathfrak{l}) \\ \in \mathscr{P}^{\mathfrak{l}-1}_{\lambda_1^c}}} h_m(\mathsf{Z}_{[m]}(t, \underline{\lambda})) \mathcal{O}_\mathfrak{l}(Z_{\lambda_1}, \cdots, Z_{\lambda_\mathfrak{l}})$$
$$\times \Delta\!\!\!\Delta^{[m]}_{|\lambda_1|}(Z_{\lambda_1}) \prod_{i=2}^{\mathfrak{l}} \Delta\!\!\!\Delta_{|\lambda_i|}(Z_{\lambda_i}) \mathbb{1}_{\text{no pathology}}\ .$$

Forgetting the pathological cases allows us to consider a simpler parametrization of the pseudotrajectory. We construct the graph G by removing the edges $(i,j)_{\tau,s}$ where i and j are in $[m]$. The edges of G can be ordered: $(i_k, j_k)_{\tau_k, s_k}$ with $\tau_1 < \tau_2 < \cdots < \tau_{n-m}$ (the τ_i are disjoint for almost all initial data).

We can completely reconstruct the pseudotrajectory by considering only the sequence $s_1, \cdots, s_{k-m}$ and the set of tagged particles $[m]$.

Definition 3.1.2 (Second definition of pseudotrajectory)**.** *Fix $m \leq n$, an initial position Z_n and parameters $(s_k)_{k \leq n-m} \in \{\pm 1\}^{n-m}$ and $\omega \subset [n]$ with $|\omega| = m$. In order to construct the pseudotrajectory $\mathsf{Z}(\tau, Z_n, \omega, (s_k)_k)$, we need an auxiliarry an auxiliary function $\iota : [0,t] \to \mathbb{N}$, which is increasing, constant by part and left-continuous function.*

At $\tau = 0$, we set $\mathsf{Z}(0, Z_n, \omega, (s_k)_k) := Z_n$ and $\iota(0) := 1$.

Suppose that the pseudo trajectory $\mathsf{Z}(\cdot, Z_n, \omega, (s_k)_k)$ and $\iota(\cdot)$ are constructed in the time interval $[0,\tau]$. At time τ particles i and j meet, i.e.

$$|\mathsf{x}_i(\tau) - \mathsf{x}_j(\tau)| = \varepsilon,\ \ (\mathsf{x}_i(\tau) - \mathsf{x}_j(\tau))\,(\mathsf{v}_i(\tau) - \mathsf{v}_j(\tau)) > 0.$$

If $(i,j) \in \omega^2$, the two particles interact and we fix $\iota(\tau^+) := \iota(\tau)$. Otherwise, we fix $\iota(\tau^+) := \iota(\tau) + 1$ and we look at $s_{\iota(\tau)}$. If $s_{\iota(\tau)} = 1$ the two particles interact: they follow on $[\tau, \tau^+]$ the dynamic

$$\begin{cases} \dot{\mathsf{x}}_i = \mathsf{v}_i,\ \dot{\mathsf{v}}_i = \dfrac{1}{\varepsilon} \mathscr{V}\left(\dfrac{\mathsf{x}_i - \mathsf{x}_j}{\varepsilon}\right), \\[2ex] \dot{\mathsf{x}}_j = \mathsf{v}_j,\ \dot{\mathsf{v}}_j = \dfrac{-1}{\varepsilon} \mathscr{V}\left(\dfrac{\mathsf{x}_i - \mathsf{x}_j}{\varepsilon}\right). \end{cases}$$

If $s_{\iota(\tau)} = -1$ the two particles ignore each other: on $[\tau, \tau^+]$ we have

$$\begin{cases} \dot{\mathsf{x}}_i = \mathsf{v}_i,\ \dot{\mathsf{v}}_i = 0, \\ \dot{\mathsf{x}}_j = \mathsf{v}_j,\ \dot{\mathsf{v}}_j = 0. \end{cases}$$

In both cases, we define $\tau^+ > \tau$ as the fist time bigger than τ such that

$$\left|\mathsf{x}_i(\tau^+) - \mathsf{x}_j(\tau^+)\right| = \varepsilon, \ \left(\mathsf{x}_i(\tau^+) - \mathsf{x}_j(\tau^+)\right)\left(\mathsf{v}_i(\tau^+) - \mathsf{v}_j(\tau^+)\right) < 0.$$

We denote $\mathscr{R}^t_{\omega,(s_k)_k} \subset \mathbb{D}^n$ the set of initial parameters such that the pseudotrajectory has a connected collision graph and has no multiple interaction, recollision nor overlap. Hence, on $\mathscr{R}^t_{\omega,(s_k)_k} \subset \mathbb{D}^n$, the previous construction has no ambiguity.

We can reconstruct the partition $(\lambda_1, \cdots, \lambda_{\mathbf{l}})$ for given $(s_i)_{ß\leq n-m}$. We define the graph G as a subgraph of the collision graph $\mathscr{G}^{[0,t]}$ by removing the edges of the form $(i,j)_{\tau,-1}$ (we keep only the interaction). The cluster λ_1 is the union of the connected components in G of the particles $[m]$. The $(\lambda_2, \cdots, \lambda_{\mathbf{l}})$ are the other connected components.

We have the following equality

$$\Phi^{0,t}_{m,n}[h_m](Z_n) = \frac{1}{(n-m)!} \sum_{(s_k)_{k\leq n-m}} \prod_{i=1}^{n-m} s_i \ \mathbb{1}_{\mathscr{R}^t_{(s_k)_k}} h_m(\mathsf{Z}_{[m]}(\tau, Z_n, [m], (s_k)_k)) \tag{3.8}$$

We denote

$$\Phi^{>,t}_{m,n} := \Phi^t_{m,n} - \Phi^{0,t}_{m,n}. \tag{3.9}$$

3.4. **Iteration of the pseudotrajectory development.** The construction of Section 3.3 is efficient over a short time interval. To raise long time result, we need to iterate these kind of pseudotrajectory representations and to compute sums of the form

$$\sum_{\underline{i}_{n_2}} \Phi^{0,\delta}_{n_1,n_2} \circ \Phi^{0,\delta}_{n_0,n_1}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}})$$

where $n_0 \leq n_1 \leq n_2$ are three integers.

Remark 3.1.1. *In the usual framework, the pseudotrajectories have a tree (see for example* [BGSRS21, BGSRS22a, LB22]*): there are more and more particles as we go backward in time. Hence, the development has naturally a semi-group structure, and it is straight-forward to continue the development.*

In the present discussion, the pseudotrajectories have a graph structure: particles do not disappear. Hence, we need to work to iterate the process.

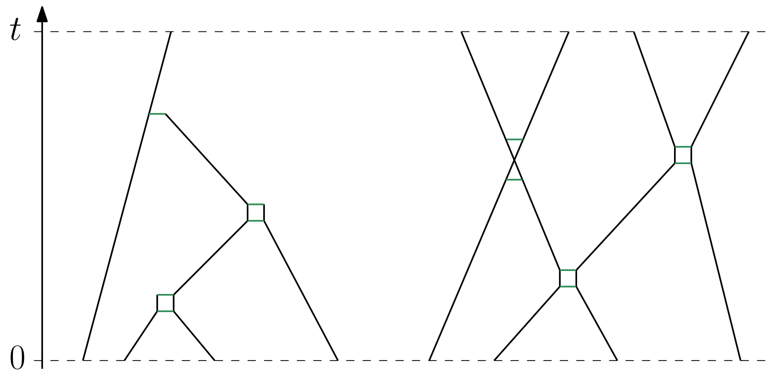

FIGURE 7. On the left a tree pseudotrajectory, on the right a graph pseudotrajectory.

We need a new definition of pseudotrajectory:

Definition 3.1.3 (Third definition of pseudotrajectory)**.** *Fix $m \leq n$. For a family of parameters $(\omega_1, \omega_2, (s_k)_{k\leq n-m})$ with $\omega_1 \subset \omega_2 \subset [n]$, $|\omega_1| = m$ and $(s_k)_{k\leq n-m} \in \{\pm 1\}^{n-m}$, we define $\mathrm{Z}(\tau, Z_n, \omega_1, \omega_2, (s_k)_k)$ as*

- *for $\tau \leq \delta$,*
$$\mathrm{Z}(\tau, Z_n, \omega_1, \omega_2, (s_k)_{k\leq n-m}) := \mathsf{Z}(\tau, Z_n, \omega_2, (s_k)_{k\leq n-|\omega_2|}),$$
- *for $\tau > \delta$,*
$$\mathrm{Z}_{\omega_2}(\tau, Z_n, \omega_1, \omega_2, (s_k)_{k\leq n-m}) := \mathsf{Z}(\tau - \delta, \mathrm{Z}_{\omega_2}(\delta), \omega_1, (s_k)_{n-|\omega_2|<k\leq n-m}),$$
and for all $i \in [n] \setminus \omega_2$
$$\mathrm{z}_i(\tau) := (\mathrm{x}_i(\delta) + (\tau - \delta)\mathrm{v}_i(\delta), \mathrm{v}_i(\delta)).$$

Remark 3.1.2. *Note that the particles in $[n_2]\setminus\omega_2$ are virtual since time δ: they do not interact with any other particle.*

We define the collision graph $\mathscr{G}^{[0,t]}$ as before. We define $G_1^{[0,\delta]}$ the subgraph of $\mathscr{G}^{[0,t]}$ with edges

$$\big\{(i,j)_\tau \in \mathscr{G}^{[0,t]},\ \tau\in[0,\delta]\big\},$$

and for $\tau\in[0,t]$ the graph $G_2^{[\tau,t]}$ the subgraph of $\mathscr{G}^{[0,t]}$ with edges

$$\big\{(i,j)_\tau \in \mathscr{G}^{[0,t]},\ \tau\in[\tau,t],\ (i,j)\in\omega_2^2\big\}.$$

An *admissible initial* data is such that $G_1^{[0,\delta]}$ and $G_2^{[\delta,t]}$ have no cycle.

Definition 3.1.4 (Semi-tree condition)**.** *Fix t and δ such that $t/\delta = K\in\mathbb{N}^*$, parameters $(\omega_1,\omega_2,(s_k)_{k\le n-m})$ and admissible initial data Z_{n_2}.*

We define for $k\in[0,K[$ the sets ϖ_k the connected component of ω_1 in the graph $G_2^{[t-k\delta,st]}$. The graph $G_2^{[\delta,t]}$ checks is a semi-tree condition *if the edges of $G_2^{[t-k\delta,t-(k-1)\delta]}$ are of the form $(i,j)_\tau$ with $(i,j)\in\varpi_k^2\setminus\varpi_{k-1}^2$.*

We denote $\mathscr{R}^t_{\omega_1,\omega_2,(s_k)_k}$ the set of admissible initial data such that $G_2^{[\delta,t]}$ verifies the semi-tree condition.

Fix $n_0\le n_1\le n_2$ and a test function h_{n_0}. We have directly

$$\Phi^{0,\delta}_{n_1,n_2}\circ\Phi^{0,\delta}_{n_0,n_1}[h_{n_0}](Z_{n_2})$$
$$=\frac{1}{(n_2-n_1)!(n_1-n_0)!}\sum_{(s_k)_{k\le n_2-n_0}}\prod_{k=1}^{n_2-n_0}s_k\,h_{n_0}\left(\mathrm{Z}_{[n_0]}(2\delta,Z_{n_2},[n_0],[n_1],(s_k)_k)\right)\mathbb{1}_{\mathscr{R}^t_{[n_0],[n_1],(s_k)_k}}.$$

and

$$\sum_{n_1=n_0}^{n_2}\frac{(n_2-n_1)!(n_1-n_0)!}{(n_2-n_0)!}\sum_{\substack{[n_0]\subset\omega\subset[n_2]\\|\omega|=n_1}}\Phi^{0,\delta}_{n_1,n_2}\circ\Phi^{0,\delta}_{n_0,n_1}[h_{n_0}](Z_{[n_1]},Z_{\omega\setminus[n_1]},Z_{[n_2]\setminus\omega})$$
$$=\frac{1}{(n_2-n_0)!}\sum_{\substack{[n_0]\subset\omega\subset[n_2]\\(s_k)_{k\le n_2-n_0}}}\prod_{k=1}^{n_2-n_0}s_k\,h_{n_0}\left(\mathrm{Z}_{[n_0]}(2\delta,Z_{n_2},[n_0],\omega,(s_k)_k)\right)\mathbb{1}_{\mathscr{R}^t_{[n_0],\omega,(s_k)_k}}.$$

Remark 3.1.3. *We have removed overlap in order to make this equality direct.*

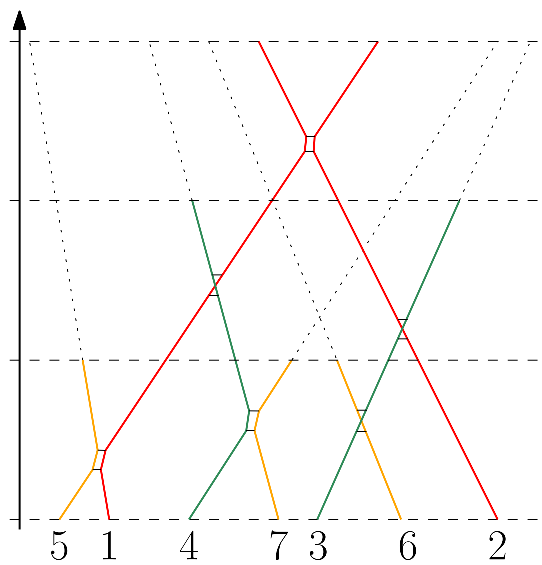

FIGURE 8. Here the pseudotrajectory checks the semi-tree condition, with $\omega_1=\{1\}$, $\varpi_1=\{1,2\}$, $\varpi_2=\{1,2,3,4\}$ and $\varpi_3=\{1,2,3,4,5,6\}$. In the picture, when the trajectory of a particle is a dotted line, it does not overlap nor interact with any other particle.

We separate $\mathscr{R}^t_{[n_0],\omega,(s_k)_k}$ into two pieces: $\mathscr{R}^{>,t}_{[n_0],\omega,(s_k)_k}$ where the collision graph $\mathscr{G}^{[0,t]}$ has at least one cycle and $\mathscr{R}^{0,t}_{[n_0],\omega,(s_k)_k}$ where $\mathscr{G}^{[0,t]}$ has no cycle.

For $Z_{n_2} \in \mathscr{R}^{0,t}_{[n_0],\omega,(s_k)_k}$, there are exactly $n_2 - n_0$ collisions in the collision graph, and ω is not needed to reconstruct the pseudotrajectory. In addition, its collision graph checks the semi-tree conditions. Fixing the parameters $(s_k)_k$, the sets $(\mathscr{R}^{0,t}_{[n_0],\omega,(s_k)_k})_\omega$ are disjoints, as ω^c is the connected components of $[n_0]$ in $G_2^{[\delta,t]}$. We denote $\mathscr{R}^{0,t}_{[n_0],(s_k)_k}$ the set of initial data whose pseudotrajectories have no recollision and check the semi-tree property:

$$\mathscr{R}^{0,t}_{[n_0],(s_k)_k} := \bigcup_{[n_0]\subset\omega\subset[n_2]} \mathscr{R}^{0,t}_{[n_0],\omega,[n_2]}.$$

We define now the functional

$$\Psi^{0,t}_{n_0,n_2}[h_{n_0}] := \frac{1}{(n_2-n_0)!} \sum_{(s_k)_{k\le n_2-n_0}} \prod_{k=1}^{n_2-n_0} s_k\, h_{n_0}\left(\mathsf{Z}_{[n_0]}(t,\cdot,[n_0],(s_k)_k)\right) \mathbb{1}_{\mathscr{R}^{0,t}_{[n_0],(s_k)_k}} \tag{3.10}$$

$$\Psi^{>,t}_{n_0,n_2}[h_{n_0}] := \frac{1}{(n_2-n_0)!} \sum_{\substack{[n_0]\subset\omega\subset[n_2]\\ (s_k)_{k\le n_2-n_0}}} \prod_{k=1}^{n_2-n_0} s_k\, h_{n_0}\left(\mathsf{Z}_{[n_0]}(t,\cdot,[n_0],\omega,(s_k)_k)\right) \mathbb{1}_{\mathscr{R}^{>,t}_{[n_0],\omega,(s_k)_k}}. \tag{3.11}$$

We obtain

$$\sum_{\underline{i}_{n_2}} \sum_{n_1=n_0}^{n_2} \Phi^{0,\delta}_{n_1,n_2} \circ \Phi^{0,\delta}_{n_0,n_1}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}) := \sum_{\underline{i}_{n_2}} \Psi^{0,2\delta}_{n_0,n_2}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}) + \sum_{\underline{i}_{n_2}} \Psi^{>,2\delta}_{n_0,n_2}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}).$$

The construction can be iterated: $\forall k\in\mathbb{N}$

$$\sum_{\underline{i}_{n_2}} \sum_{n_1=n_0}^{n_2} \Psi^{0,k\delta}_{n_1,n_2} \circ \Phi^{0,\delta}_{n_0,n_1}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}) := \sum_{\underline{i}_{n_2}} \Psi^{0,(k+1)\delta}_{n_0,n_2}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}) + \sum_{\underline{i}_{n_2}} \Psi^{>,(k+1)\delta}_{n_0,n_2}[h_{n_0}](\mathbf{Z}_{\underline{i}_{n_2}}).$$

The functional $\Psi^{0,t}_{m,n}$ are introduced to implement the sampling: for $t>2\delta$ and $\mathbf{Z}_{\mathscr{N}}\in\Upsilon_\varepsilon$

$$\begin{aligned}
\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t)) &= \sum_{n\ge1}\sum_{\underline{i}_n} \Phi^{0,\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\delta)) + \sum_{n\ge1}\sum_{\underline{i}_n} \Phi^{>,\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\delta))\\
&= \sum_{n'\ge n\ge 0}\sum_{\underline{i}_{n'}} \Big(\Phi^{0,\delta}_{n,n'}\Psi^{0,\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-2\delta)) + \Phi^{>,\delta}_{n,n'}\Psi^{0,\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-k\delta))\Big)\\
&\qquad\qquad + \sum_{n\ge1}\sum_{\underline{i}_n} \Phi^{>,\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\delta))\\
&= \sum_{n\ge1}\sum_{\underline{i}_n} \Psi^{0,t}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(0)) + \sum_{k=1}^{2}\sum_{n\ge1}\sum_{\underline{i}_n} \Psi^{>,k\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-k\delta))\\
&\qquad + \sum_{k=1}^{2}\sum_{1\le n\le n'}\sum_{\underline{i}_{n'}} \Phi^{>,\delta}_{n,n'}\circ\Psi^{0,(k-1)\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-k\delta)).
\end{aligned}$$

The preceding computation can be iterated: for some time t, $\theta<t$ and δ such that $\theta/\delta=K\in\mathbb{N}$, and any initial data $\mathbf{Z}_{\mathscr{N}}\in\Upsilon_\varepsilon$

$$\begin{aligned}
\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t)) = \sum_{n\ge1}\sum_{\underline{i}_n} \Psi^{0,\theta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\theta)) &+ \sum_{k=1}^{K}\sum_{n\ge1}\sum_{\underline{i}_n} \Psi^{>,k\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-k\delta))\\
&+ \sum_{k=1}^{K}\sum_{1\le n\le n'}\sum_{\underline{i}_{n'}} \Phi^{>,\delta}_{n,n'}\circ\Psi^{0,(k-1)\delta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-k\delta)).
\end{aligned} \tag{3.12}$$

3.5. **The decomposition of the covariance.** The final ingredient is a second sampling on a longer time scale $\theta = 1/\beta\log|\log\varepsilon|$ which control the growth of the number of collisions.

Definition 3.1.5 (Number of particles at time τ)**.** *Fix t and δ such that $t/\delta=K\in\mathbb{N}^*$, parameters $(\{1\},\omega_2,(s_k)_{k\le n-m})$ and admissible initial data $Z_{n_2}\in\mathscr{R}^{0,t}_{\{1\},\omega_2,(s_k)_k}$. For $\tau=k\delta$, the* number of particles *at time τ, $\mathfrak{n}(\tau)$, is defined as the size of the connected component of $\{1\}$ in $G_2^{[\tau,t]}$.*

We want that the number of particles $(\mathfrak{n}(t-k\theta))$ grows at most exponentially.

Fix $1 \leq n_1 \leq \cdots \leq n_l$. We denote $\underline{n}_l := (n_i)_{i\leq l}$. For $t \in ((l-1)\theta, l\theta]$

$$\Psi^{0,t}_{\underline{n}_l}[h] := \frac{1}{(n_l-1)!} \sum_{(s_k)_{k\leq n_l-1}} \prod_{k=1}^{n_l-1} s_k h(\mathrm{z}_1(t,\cdot,\{1\},(s_k)_k)) \mathbb{1}_{\mathscr{R}^{0,t}_{\{1\},(s_k)_k}} \prod_{i=1}^{\lfloor t/\theta \rfloor} \mathbb{1}_{\mathfrak{n}(t-i\theta)=n_i}, \tag{3.13}$$

and for $t \in [(l-2)\theta, (l-1)\theta]$

$$\Psi^{>,t}_{\underline{n}_l}[h] := \frac{1}{(n_l-1)!} \sum_{\substack{1\in\omega\subset[n_l]\\(s_k)_{k\leq n_l-1}}} \prod_{k=1}^{n_l-1} s_k\, h(\mathrm{z}_1(t,\cdot,\{1\},\omega,(s_k)_k)) \mathbb{1}_{\mathscr{R}^{>,t}_{\{1\},\omega,(s_k)_k}} \prod_{i=1}^{l-2} \mathbb{1}_{\mathfrak{n}(t-i\theta)=n_i}. \tag{3.14}$$

We can iterate the preceding decomposition of $\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i)(t))$:

$$\begin{aligned}
\sum_{n\geq 1}\sum_{\underline{i}_n} \Psi^{0,\theta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\theta))
&= \sum_{n'\geq n}\sum_{\underline{i}_{n'}} \Phi^{0,\delta}_{n,n'}\circ\Psi^{0,\theta}_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(t-\theta) + \sum_{n'\geq n}\sum_{\underline{i}_{n'}} \Phi^{>,\delta}_{n,n'}\circ\Psi^{0,\theta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-k\delta))\\
&= \sum_{n'\geq n}\sum_{\underline{i}_{n'}} \Phi^{0,\delta}_{(1,n,n')}[h](\mathbf{Z}_{\underline{i}_n}(t-\theta) + \sum_{n'\geq n}\sum_{\underline{i}_{n'}} \Phi^{0,\delta}_{(1,n,n')}[h](\mathbf{Z}_{\underline{i}_n}(t-\theta)\\
&\qquad + \sum_{1\leq n\leq n'}\sum_{\underline{i}_{n'}} \Phi^{>,\delta}_{n,n'}\circ\Psi^{0,\theta}_{1,n}[h](\mathbf{Z}_{\underline{i}_{n'}}(t-k\delta))
\end{aligned} \tag{3.15}$$

The decomposition is performed until reaching the time 0: denoting $K := t/\theta \in \mathbb{N}$ and $K' := \theta/\delta \in \mathbb{N}$, for almost any initial data $\mathbf{Z}_{\mathscr{N}}(0) \in \mathscr{D}$,

$$\sum_{i=1}^{\mathscr{N}} h(\mathbf{z}_i(t)) = \sum_{\substack{(n_j)_{j\leq K}\\0\leq n_j-n_{j-1}\leq 2^j}} \sum_{\underline{i}_{n_k}} \Psi^{0,k\theta}_{\underline{n}_k}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right) \tag{3.16}$$

$$+ \sum_{1\leq k\leq K} \sum_{\substack{(n_j)_{j\leq k-1}\\0\leq n_j-n_{j-1}\leq 2^j}} \sum_{n_k\geq 2^k+n_{k-1}} \sum_{\underline{i}_{n_k}} \Psi^{0,k\theta}_{\underline{n}_k}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right) \tag{3.17}$$

$$+ \sum_{\substack{0\leq k\leq K-1\\1\leq k'\leq K'}} \sum_{\substack{(n_j)_{j\leq k}\\0\leq n_j-n_{j-1}\leq 2^j}} \sum_{n_{k+2}\geq n_{k+1}\geq n_k} \sum_{\underline{i}_{n_k}} \Psi^{>,t-t_s}_{\underline{n}_{k+1}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t_s)\right) \tag{3.18}$$

$$+ \sum_{\substack{0\leq k\leq K-1\\1\leq k'\leq K'}} \sum_{\substack{(n_j)_{j\leq k}\\0\leq n_j-n_{j-1}\leq 2^j}} \sum_{\substack{n_{k+1}\geq n_k\\n_{k+2}\geq n_{k+1}}} \sum_{\underline{i}_{n_{k+2}}} \Psi^{0,t_s-\delta}_{\underline{n}_{k+1}}\Phi^{0,\delta}_{n_{k+1},n_{k+2}}[h]\left(\mathbf{Z}_{\underline{i}_{n_{k+2}}}(t_s)\right) \tag{3.19}$$

where $t_s := t-k\theta-k'\delta$.

Finally, the convariance is split into five parts

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0(g)\right] = G^{\mathrm{main}}_\varepsilon(t) + G^{\mathrm{clus}}_\varepsilon(t) + G^{\mathrm{exp}}_\varepsilon(t) + G^{\mathrm{rec},1}_\varepsilon(t) + G^{\mathrm{rec},2}_\varepsilon(t) \tag{3.20}$$

- where the main part

$$G^{\mathrm{main}}_\varepsilon(t) := \sum_{\substack{(n_j)_{j\leq K}\\0\leq n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_K}} \Psi^{0,t}_{\underline{n}_K}[h]\Big(\mathbf{Z}_{\underline{i}_{n_K}}(0)\Big)\Big(\mathbf{Z}_{\underline{i}_{n_K}}(0)\Big)\zeta^0_\varepsilon(g)\right], \tag{3.21}$$

- the first error due to the symmetric conditioning and the suppression of the overlaps

$$G^{\mathrm{clus}}_\varepsilon(t) := \mathbb{E}_\varepsilon\Big[\zeta^t_\varepsilon(h)\zeta^0(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\Big] - \sum_{\substack{(n_j)_{j\leq K}\\0\leq n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_K}} \Psi^{0,t}_{\underline{n}_K}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right], \tag{3.22}$$

- the part controlling the growth of the number of particle,

$$G^{\mathrm{exp}}_\varepsilon(t) := \mathbb{E}_\varepsilon\left[(3.17)\times\frac{1}{\sqrt{\mu}}\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right], \tag{3.23}$$

- the part corresponding to non-local recollision,

$$G_\varepsilon^{\mathrm{rec},1}(t) := \mathbb{E}_\varepsilon\left[(3.18) \times \frac{1}{\sqrt{\mu}}\zeta_\varepsilon^0(g)\mathbb{1}_{\Upsilon_\varepsilon}\right], \tag{3.24}$$

- and the part corresponding to local recollision

$$G_\varepsilon^{\mathrm{rec},2}(t) := \mathbb{E}_\varepsilon\left[(3.19) \times \frac{1}{\sqrt{\mu}}\zeta_\varepsilon^0(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]. \tag{3.25}$$

The parts $G_\varepsilon^{\mathrm{clus}}(t)$ and $G_\varepsilon^{\mathrm{exp}}(t)$ are estimated by (5.1):

$$|G_\varepsilon^{\mathrm{exp}}(t) + G_\varepsilon^{\mathrm{rec}}(t)| \leq C\|g\|_0\|h\|_0\left(\varepsilon^{1/3}(C\tfrac{t}{\mathfrak{d}})^{2^K} + \tfrac{t\theta^{1/2}}{\mathfrak{d}^{3/2}}\right),$$

the part $G_\varepsilon^{\mathrm{rec},1}(t)$ is estimated by (7.1)

$$\left|G_\varepsilon^{\mathrm{rec},1}(t)\right| \leq \|g\|_0\|h\|_0\varepsilon^{\mathfrak{a}/2}K2^{K^2}(Ct)^{2^K+2d+6},$$

the part $G_\varepsilon^{\mathrm{rec},2}(t)$ is bounded at (8.1):

$$\left|G_\varepsilon^{\mathrm{rec},2}(t)\right| \leq C\|h\|_0\|g\|_0K2^{K^2}(C\tfrac{t}{\mathfrak{d}})^{2^{K+1}}\varepsilon^{\frac{\mathfrak{a}}{2}},$$

and the convergence of $G_\varepsilon^{\mathrm{main}}(t)$ is given by (6.20):

$$G_\varepsilon^{\mathrm{main}}(t) = \int_{\mathbb{D}} h(z)\mathbf{g}_\alpha(t,z)M(z)dz + O\left(\left(C\tfrac{\theta t}{\mathfrak{d}^2} + \varepsilon^{\mathfrak{a}}K2^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\right)\|h\|_1\|g\|_1\right)$$

where $\mathbf{g}_\alpha(t,z)$ is the solution of the linearized Boltzmann equation (2.16). Combining these four estimations, we obtain the expected bound (2.15)

$$\mathbb{E}_\varepsilon\left[\zeta_\varepsilon^t(g)\zeta_\varepsilon^0(h)\right] = \int_{\mathbb{D}} h(z)\mathbf{g}_\alpha(t,z)M(z)dz + O\left(\left(C\tfrac{\theta t}{\mathfrak{d}^2} + \varepsilon^{\mathfrak{a}}K2^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\right)\|h\|_1\|g\|_1\right) \tag{3.26}$$

Remark 3.1.4. *In this section, we have defined three different pseudotrajectories :*

- *in Definition 3.0.3 we have defined the general definition of pseudotrajectory, which is used in the estimation of pathological recollision* $G_\varepsilon^{rec,2}(t)$*,*
- *the pseudotrajectories of Definition 3.1.2 have no recollision and are used to treat* $G_\varepsilon^{main}(t)$*,* $G_\varepsilon^{clust}(t)$ *and* $G_\varepsilon^{exp}(t)$*,*
- *Definition 3.1.3 describes pseudotrajectories with nonpathological recollision. There are used to bound* $G_\varepsilon^{rec,2}(t)$*.*

4. Quasi-orthogonality estimates

The different error terms obtained in the previous section are of the form

$$\mathbb{E}_\varepsilon\left[\sum_{\underline{i}_n}\Phi_n[h](\mathbf{Z}_{\underline{i}_n}(t_s))\zeta_\varepsilon^0(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]$$

where the $\Phi_n : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D}^n)$ are continuous functionals. In order to bound the errors, we need an $L^2(\mathbb{P}_\varepsilon)$ bound of

$$\hat{\Phi}_n = \frac{1}{\mu^n}\sum_{\underline{i}_n}\Phi_n[h](\mathbf{Z}_{\underline{i}_n}) - \mathbb{E}[\Phi_n].$$

The following section is dedicated to the derivation of such estim, using detailed estimations on the functionals $\Phi_n[h]$. We will use, in particular, that we can bound the $\Phi_n[h](Z_n)$ by looking only at the relative positions of particles inside Z_n.

Definition 4.0.1. *We denote for* $y \in \mathbb{T}$ *the translation operator*

$$\mathrm{tr}_y : \begin{cases} \mathbb{D}^n \to \mathbb{D}^n \\ (X_n, V_n) \mapsto (x_1 + y, \cdots, x_n + y, V_n) \end{cases}. \tag{4.1}$$

Fix n,m two integers, $g_n:\mathbb{D}^n\to\mathbb{R}$, $h_m:\mathbb{D}^m\to\mathbb{R}$ two functions and $l\in[0,\min(n,m)]$. We define the multiplication on l variable $\circledast_l$ as

$$\begin{aligned}(4.2)\quad & g_n\circledast_l h_m(Z_{n+m-l})\\ &:=\frac{1}{(n+m-l)!n!m!}\sum_{\substack{\sigma\in\mathfrak{S}([n+m-l])\\ \sigma'\in\mathfrak{S}([1,n])\\ \sigma''\in\mathfrak{S}([n-l,n+m-l])}} g_n(Z_{\sigma\sigma'([1,n])})h_m(Z_{\sigma\sigma''([n+1-l,n+m-l])}).\end{aligned}$$

where $\mathfrak{S}(\omega)$ is the set of permutation of ω.

Theorem 7. *Fix $m<n$ two positive integers, and $g_n:\mathbb{D}^n\to\mathbb{R}$, $h_m\mathbb{D}^m\to\mathbb{R}$ two functions such that there exists a finite sequence $(c_0,c_0',c_1,\cdots,c_n)\in\mathbb{R}_+^{n+2}$ bounding g_n, h_m in the following way:*

$$\int_{x_1=0}\sup_{y\in\mathbb{T}}\big|g_n\big(\operatorname{tr}_y Z_n\big)\big|\frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}}\,\mathrm{d}X_{2,n}\,\mathrm{d}V_n\le c_0, \tag{4.3}$$

$$\int_{x_1=0}\sup_{y\in\mathbb{T}}\big|h_m\big(\operatorname{tr}_y Z_m\big)\big|\frac{e^{-\mathscr{H}_m(Z_m)}}{(2\pi)^{\frac{md}{2}}}\,\mathrm{d}X_{2,m}\,\mathrm{d}V_m\le c_0' \tag{4.4}$$

and for all $l\in[1,m]$

$$\int_{x_1=0}\sup_{y\in\mathbb{T}}\big|g_n\circledast_l h_m\big(\operatorname{tr}_y Z_{n+m-l}\big)\big|\frac{e^{-\mathscr{H}_{n+m-l}(Z_{n+m-l})}}{(2\pi)^{\frac{(n+m-l)d}{2}}}\,\mathrm{d}X_{2,n+m-l}\,\mathrm{d}V_{n+m-l}\le\frac{\mu^{l-1}}{n^l}c_l. \tag{4.5}$$

There exists a constant $C>0$ depending only on the dimension such that

$$\big|\mathbb{E}_\varepsilon\big[g_n\big]\big|\le C^n c_0,\ \ \big|\mathbb{E}_\varepsilon\big[h_m\big]\big|\le C^m c_0' \tag{4.6}$$

and

$$\mathbb{E}_\varepsilon\Big[\mu\hat{g}_n\hat{h}_m\Big]=\sum_{l=1}^m\binom{n}{l}\binom{m}{l}\frac{l!}{\mu^{l-1}}\mathbb{E}_\varepsilon\big[g_n\circledast_l h_m\big]+O\left(C^{n+m}c_0c_0'\,\frac{\varepsilon}{\mathfrak{d}}\right). \tag{4.7}$$

In particular

$$\Big|\mathbb{E}_\varepsilon\big[\mu\hat{g}_n\hat{h}_m\big]\Big|\le C^{n+m}\left(\max_{1\le l\le m}c_l+c_0c_0'\,\frac{\varepsilon}{\mathfrak{d}}\right). \tag{4.8}$$

Proof of Theorem 7.

- We begin by the proof of (4.6).

Using invariance under permutation,

$$\begin{aligned}\mathbb{E}_\varepsilon[g_n]&=\frac{1}{\mathscr{Z}\mu^n}\sum_{p\ge n}\frac{\mu^p}{p!}\int\sum_{\substack{(i_1,\cdots i_n)\\ \forall k,i_k\le p}}g_n(Z_n)e^{-\mathscr{H}_p(Z_p)}\frac{\mathrm{d}Z_p}{(2\pi)^{dp/2}}\\ &=\frac{1}{\mathscr{Z}\mu^n}\sum_{p\ge n}\frac{\mu^p}{p!}\frac{p!}{(n-p!)}\int g_n(Z_n)e^{-\mathscr{H}_p(Z_p)}\frac{\mathrm{d}Z_p}{(2\pi)^{dp/2}}\\ &=\frac{1}{\mathscr{Z}}\sum_{p\ge0}\frac{\mu^p}{p!}\int g_n(Z_n)e^{-\mathscr{V}_{n+p}(X_n,\underline{X}_p)}M^{\otimes n}\,\mathrm{d}Z_n\,\mathrm{d}\underline{X}_p.\end{aligned}$$

We recall the notation

$$\mathscr{V}_n(X_n):=\alpha\sum_{1\le i<j\le n}\mathscr{V}\left(\frac{x_i-x_j}{\varepsilon}\right)$$

and we denote in the following $\Omega:=\{X_n,\underline{x}_1,\cdots,\underline{x}_p\}$ and for $X,Y\in\Omega$,

$$\varphi(\underline{x}_i,\underline{x}_j):=\exp\left(-\alpha\mathscr{V}\left(\frac{\underline{x}_i-\underline{x}_j}{\varepsilon}\right)\right)-1,\ \ \varphi(X_N,\underline{x}_j):=\exp\left(-\alpha\sum_{i=1}^N\mathscr{V}\left(\frac{x_i-\underline{x}_j}{\varepsilon}\right)\right)-1. \tag{4.9}$$

Defining $d((x_1,\cdots,x_n),(y_1,\cdots,y_m))$ as the minimum of the $|x_i-y_j|$, we can bound φ by

$$-\mathbb{1}_{d(X,Y)<\varepsilon}\le\varphi(X,Y)\le0.$$

We decompose $\exp\big(-\mathscr{V}_{n+p}(X_{n+1},\underline{X}_p)\big)$

$$e^{-\mathscr{V}_{n+p}(X_{n+1},\underline{X}_p)} = e^{-\mathscr{V}^\varepsilon_n(X_n)} \prod_{\substack{(X,Y)\in\Omega^2\\ X\neq Y}} (1+\varphi(X,Y)) = e^{-\mathscr{V}_n(X_n)} \sum_{G\in\mathscr{G}(\Omega)} \prod_{(X,Y)\in E(G)} \varphi(X,Y)$$

where $\mathscr{G}(\Omega)$ is the set of non oriented graphs on Ω and $E(G)$ the set of edges of G. Denoting by $\mathscr{C}(\omega)$ the set of connected graphs on ω,

$$\begin{aligned}
&\exp\big(-\mathscr{V}_{n+p}(X_n,\underline{X}_p)\big)\\
&= \sum_{\omega\subset[1,p]} \Bigg(e^{-\mathscr{V}_n(X_n)} \sum_{G\in\mathscr{C}(\omega\cup\{X_n\})} \prod_{(X,Y)\in E(G)} \varphi(X,Y) \sum_{G\in\mathscr{G}(\omega^c)} \prod_{(X,Y)\in E(G)} \varphi(X,Y)\Bigg)\\
&= e^{-\mathscr{V}_n(X_n)} \sum_{\omega\subset[1,p]} \Bigg(e^{-\mathscr{V}_{|\omega^c|}(\underline{X}_{\omega^c})} \sum_{G\in\mathscr{C}(\omega\cup\{X_n\})} \prod_{(X,Y)\in E(G)} \varphi(X,Y)\Bigg)\\
&=: e^{-\mathscr{V}_n(X_n)} \sum_{\omega\subset[1,p]} e^{-\mathscr{V}_{|\omega^c|}(\underline{X}_{\omega^c})} \psi^n_{|\omega|}(X_n,\underline{X}_\omega)\,.
\end{aligned} \tag{4.10}$$

Thus, using exchangeability, $\mathbb{E}_\varepsilon[g_n]$ is equal to

$$\begin{aligned}
&\frac{1}{\mathscr{Z}} \sum_{p\geq 0} \sum_{p_1+p_2=p} \frac{\mu^p}{p!} \frac{p!}{p_1!p_2!} \int g_n(Z_n) \psi^n_{p_1}(X_n,\underline{X}_{p_1}) e^{-\mathscr{V}_{p_2}(\underline{X}'_{p_2})} \frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}} \,\mathrm{d}Z_n\,\mathrm{d}\underline{X}_{p_1}\,\mathrm{d}\underline{X}'_{p_2}\\
&= \left(\frac{1}{\mathscr{Z}} \sum_{p\geq 0} \frac{\mu^p}{p!} \int e^{-\mathscr{V}_p(\underline{X}_p)}\,\mathrm{d}\underline{X}_p \right) \left(\sum_{p\geq 0} \frac{\mu^p}{p!} \int g_n(Z_n)\psi^n_p(X_n,\underline{X}_p) \frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}} \,\mathrm{d}Z_n\,\mathrm{d}\underline{X}_p \right)\\
&= \sum_{p\geq 0} \frac{\mu^p}{p!} \int g_n(Z_n)\psi^n_p(X_n,\underline{X}_p) \frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}} \,\mathrm{d}Z_n\,\mathrm{d}\underline{X}_p.
\end{aligned} \tag{4.11}$$

We recall Penrose's tree inequality (see [Pen63, BGSRS22c, Jan]), for function φ define in (4.9),

$$\begin{aligned}
\left| \sum_{C\in\mathscr{C}(\Omega)} \prod_{(X,Y)\in E(C)} \varphi(X,Y) \right| &\leq \sum_{T\in\mathscr{T}(\Omega)} \prod_{(X,Y)\in E(T)} |\varphi(X,Y)|\\
&\leq \sum_{T\in\mathscr{T}(\Omega)} \prod_{(X,Y)\in E(T)} \mathbb{1}_{d(X,Y)<\varepsilon}
\end{aligned} \tag{4.12}$$

with $\mathscr{T}(\Omega)$ the set of trees (minimally connected graphs) on Ω. Fix $\mathrm{tr}_{-x_1} X_n$ (the relative position between particles $1,\cdots,n$). Integrating a constraint $\varphi(\underline{x}_i,\underline{x}_j)$ provides a factor $\mathbf{c}_d\varepsilon^d$, $\varphi(X_n,\underline{x}_j)$ a factor $n\mathbf{c}_d\varepsilon^d$ (where $\mathbf{c}_d$ is the volume of a sphere of diameter 1). As there are (see for example the Section 2 of [BGSRS22c] or [Jan])

$$\frac{(p-1)!}{(d_0-1)!(d_1-1)!\cdots(d_p-1)!}$$

trees with specified vertex degrees $d_0,\cdots,d_p$ associated to vertices X_n, $\underline{x}_1,\cdots,\underline{x}_p$, we get

$$\begin{aligned}
\left| \int \psi^n_p(X_n\underline{X}_p)\,\mathrm{d}\underline{X}_p \right| &\leq \sum_{\substack{d_1,\cdots,d_p\geq 1\\ d_0+\cdots+d_p=2p}} \frac{(p-1)!}{(d_0-1)!(d_1-1)!\cdots(d_p-1)!} n^{d_0} (\mathbf{c_d}\varepsilon^d)^p\\
&\leq (p-1)!(\mathbf{c_d}\varepsilon^d)^p \left(\sum_{d_0\geq 1} \frac{n^{d_0}}{(d_0-1)!} \right) \left(\sum_{d_1\geq 1} \frac{1}{(d_1-1)!} \right) \cdots \left(\sum_{d_p\geq 1} \frac{1}{(d_p-1)!} \right)\\
&\leq (p-1)!\, ne^n \big(e\mathbf{c_d}\varepsilon^d\big)^p.
\end{aligned} \tag{4.13}$$

We can integrate on the rest of parameters using (4.3). Hence

$$|\mathbb{E}_\varepsilon[g_n]| \leq \sum_{p\geq 0} \frac{(p-1)!\, ne^n \big(e\mathbf{c_d}\mu\varepsilon^d\big)^p}{p!} \int |g_n(Z_n)| \frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}} \,\mathrm{d}Z_n \leq c_0 \sum_{p\geq 0} C^n (C\varepsilon/\mathfrak{d})^p.$$

The series converge for ε small enough as $\mathfrak{d} \gg \varepsilon$. This concludes the proof of (4.6).

- We treat now (4.7). Recall first that

$$\mathbb{E}_\varepsilon\left[\mu\hat{g}_n\hat{h}_m\right] = \frac{1}{\mu^{n+m-1}}\mathbb{E}_\varepsilon\left[\sum_{\underline{i}_n} g_n(\mathbf{Z}_{\underline{i}_n})\sum_{\underline{j}_m} h_m(\mathbf{Z}_{\underline{j}_m})\right] - \mu\mathbb{E}_\varepsilon\left[g_n\right]\mathbb{E}_\varepsilon\left[h_m\right].$$

Let us count the number of ways such that $\underline{i}_n$ and $\underline{j}_m$ can intersect on a set of length l. We have to choose two sets $A \subset [n]$ and $A' \subset [m]$ of length l, and a bijection $\sigma : A \to A'$ such that for all indices $k \in A$, $i_k = j_{\sigma k}$ and that $\underline{i}_{A^c}$ does not interesect $\underline{j}_{(A^c)'}$. Thus, using the symmetry,

$$\begin{aligned}\mathbb{E}_\varepsilon\left[\mu\,\hat{g}_n\hat{h}_m\right] =& \sum_{l=1}^m \binom{n}{l}\binom{m}{l}\frac{l!}{\mu^{l-1}}\mathbb{E}_\varepsilon\left[g_n \circledast_l h_m\right] \\ &+ \mu\left(\mathbb{E}_\varepsilon\left[\frac{1}{\mu^{n+m}}\sum_{\underline{i}_{n+m}} g_n(\mathbf{Z}_{\underline{i}_n})h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}})\right] - \mathbb{E}_\varepsilon\left[g_n\right]\mathbb{E}_\varepsilon\left[g\right]\right).\end{aligned}$$

To estimate the error term in (4.7), we need to compute

$$\begin{aligned}&\mathbb{E}_\varepsilon\left[\frac{1}{\mu^{n+m}}\sum_{\underline{i}_{n+m}} g_n(\mathbf{Z}_{\underline{i}_n})h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}})\right] \\ &= \frac{1}{\mathscr{Z}}\sum_{p\geq 0}\frac{\mu^p}{p!}\int g_n(Z_n)h_m(Z'_m)\exp\left(-\mathscr{V}_{n+m+p}(X_n, X'_m, \underline{X}_p)\right) M^{\otimes n}\,\mathrm{d}Z_n M^{\otimes m}\,\mathrm{d}Z'_m\,\mathrm{d}\underline{X}_p.\end{aligned}$$

We denote in the following $\Omega := \{X_n, X'_m, \underline{x}_1, \cdots, \underline{x}_p\}$, and we decompose

$$\begin{aligned}\exp\left(-\mathscr{V}_{n+m+p}(X_n, X'_m, \underline{X}_p)\right) &= e^{-\mathscr{V}_n(X_n)}e^{-\mathscr{V}_m(X'_m)}\prod_{\substack{(X,Y)\in\Omega^2\\ X\neq Y}}(1+\varphi(X,Y)) \\ &= e^{-\mathscr{V}_n(X_n)}e^{-\mathscr{V}_m(X'_m)}\sum_{G\in\mathscr{G}(\Omega)}\prod_{(X,Y)\in E(G)}\varphi(X,Y)\end{aligned}$$

where

$$\varphi(X_n, X'_m) := \exp\left(-\alpha\sum_{i=1}^n\sum_{j=1}^m \mathscr{V}\left(\frac{x_i - x'_j}{\varepsilon}\right)\right).$$

We make a partition depending on the connected components of X_n and X'_m in G,

$$\begin{aligned}\exp\left(-\mathscr{V}_{n+m+p}(X_n, X'_m, \underline{X}_p) + \mathscr{V}_n(X_n) + \mathscr{V}_m(X'_m)\right)\right) =& \sum_{\omega\subset[1,p]}\psi^{n,m}_{|\omega|}(X_n, X'_m, \underline{X}_\omega)e^{-\mathscr{V}_{||\omega^c|}(\underline{X}_{\omega^c})} \\ &+ \sum_{\substack{\omega_1,\omega_1\subset[1,p]\\ \omega_1\cap\omega_2=\emptyset}}\psi^n_{|\omega_1|}(X_n, \underline{X}_{\omega_1})\psi^m_{|\omega_2|}(X'_m, \underline{X}_{\omega_2})e^{-\mathscr{V}_{|(\omega_1\cup\omega_2)^c|}(\underline{X}_{(\omega_1\cup\omega_2)^c})}.\end{aligned}$$

where the first line corresponds to X_n and X'_m in the same connected components, the second correspond to X_n and X'_m in disjoint connected components. In the preceding equation, we denote

$$\psi^{n,m}_{|\omega|}(X_n, X'_m, \underline{X}_\omega) = \sum_{\substack{G\in\mathscr{C}(\omega\cup\\ \{X_n,X'_m\})}}\prod_{(X,Y)\in E(G)}\varphi(X,Y).$$

Permutating the indices and using (4.11), we obtain the following equality

$$\begin{aligned}
&\frac{1}{\mathscr{Z}}\sum_{p\geq 0}\frac{\mu^p}{p!}\int g_n(Z_n)h_m(Z'_m)\sum_{\substack{\omega_1,\omega_1\subset[1,p]\\ \omega_1\cap\omega_2=\emptyset}}\psi^n_{|\omega_1|}(X_n,\underline{X}_{\omega_1})\psi^m_{|\omega_2|}(X'_m,\underline{X}_{\omega_2})e^{-\mathscr{V}_{|(\omega_1\cup\omega_2)^c|}(\underline{X}_{(\omega_1\cup\omega_2)^c})}\\
&\qquad\qquad\times\frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}}\,\mathrm{d}Z_n\frac{e^{-\mathscr{H}_m(Z'_m)}}{(2\pi)^{\frac{md}{2}}}\,\mathrm{d}Z'_m\,\mathrm{d}\underline{X}_p\\
&=\frac{1}{\mathscr{Z}}\sum_{p\geq 0}\sum_{p_1+p_2+p_3=p}\frac{\mu^p}{p!}\frac{p!}{p_1!p_2!p_3!}\int g_n(Z_n)h_{n'}(Z'_{n'})\psi^n_{p_1}(X_n,\underline{X}_{p_1})\psi^1_{p_2}(x_{n+1},\underline{X}'_{p_2})\\
&\qquad\times\Big(\frac{e^{-\mathscr{H}_n(Z_n)}}{(2\pi)^{\frac{nd}{2}}}\,\mathrm{d}Z_n d\underline{X}_{p_1}\Big)\Big(\frac{e^{-\mathscr{H}_m(Z'_m)}}{(2\pi)^{\frac{md}{2}}}\,\mathrm{d}Z'_{n'}\,\mathrm{d}\underline{X}'_{p_2}\Big)\Big(e^{-\mathscr{V}_{p_3}(\underline{X}''_{p_3})}\,\mathrm{d}\underline{X}''_{p_3}\Big)\\
&=\mathbb{E}_\varepsilon[g_n]\mathbb{E}_\varepsilon[h_{n'}],
\end{aligned}$$

and in the same way

$$\begin{aligned}
&\frac{1}{\mathscr{Z}}\sum_{p\geq 0}\frac{\mu^p}{p!}\int g_n(Z_n)h_m(Z'_m)\sum_{\omega\subset[1,p]}\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_\omega)e^{-\mathscr{V}_{||\omega^c|}(\underline{X}_{\omega^c})}\\
&\qquad\qquad\times\frac{e^{-\mathscr{H}_n(Z_n)-\mathscr{H}_m(Z'_m)}}{(2\pi)^{\frac{(n+m)d}{2}}}\,\mathrm{d}Z_n\,\mathrm{d}Z'_m\,\mathrm{d}\underline{X}_p\\
&=\frac{1}{\mathscr{Z}}\sum_{p\geq 0}\sum_{p_1+p_2=p}\frac{\mu^p}{p!}\frac{p!}{p_1!p_2!}\int g_n(Z_n)h_m(Z'_m)\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})e^{-\mathscr{V}_{p_2}(\underline{X}'_{p_2})}\\
&\qquad\qquad\times\frac{e^{-\mathscr{H}_n(Z_n)-\mathscr{H}_m(Z'_m)}}{(2\pi)^{\frac{(n+m)d}{2}}}\,\mathrm{d}Z_n\,\mathrm{d}Z'_m\,\mathrm{d}\underline{X}_{p_1}\,\mathrm{d}\underline{X}'_{p_2}\\
&=\sum_{p_1\geq 0}\frac{\mu^p}{p_1!}\int g_n(Z_n)h_m(Z'_m)\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})\frac{e^{-\mathscr{H}_n(Z_n)-\mathscr{H}_m(Z'_m)}}{(2\pi)^{\frac{(n+m)d}{2}}}\,\mathrm{d}Z_n\,\mathrm{d}Z'_m\,\mathrm{d}\underline{X}_{p_1}\,\mathrm{d}\underline{X}'_{p_2}.
\end{aligned}$$

Using again Penrose tree inequality,

$$\left|\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})\right|\leq\sum_{T\in\mathscr{T}(\Omega)}\prod_{(X,Y)\in E(T)}|\varphi(X,Y)|\leq\sum_{T\in\mathscr{T}(\Omega)}\prod_{(X,Y)\in E(T)}\mathbb{1}_{d(X,Y)<\varepsilon}. \tag{4.14}$$

First, we fix $\mathrm{tr}_{-x_1}X_n$ and $\mathrm{tr}_{-x'_1}X'_m$. Integrating a constraint $\varphi(\underline{x}_i,\underline{x}_j)$ provides a factor $\mathbf{c}_d\varepsilon^d$, $\varphi(X_n,\underline{x}_j)$ a factor $n\mathbf{c}_d\varepsilon^d$, $\varphi(X'_m,\underline{x}_j)$ a factor $m\mathbf{c}_d\varepsilon^d$ and $\varphi(X_n,X'_m)$ a factor $nm\mathbf{c}_d\varepsilon^d$. Denoting $d_0,d'_0,d_1\cdots,d_p$ the degrees of X_n, X'_m, $\underline{x}_1,\cdots,\underline{x}_m$ and $\hat{x}_1:=x_1-x'_1$,

$$\begin{aligned}
&\left|\int\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})d\underline{X}_pd\hat{x}_1\right|\\
&\qquad\leq\sum_{\substack{d'_0,d_0,\cdots,d_p\geq 1\\ d'_0+d_0+\cdots+d_p=2p}}\frac{p!}{(d'_0-1)(d_0-1)!\cdots(d_p-1)!}n^{d_0}m^{d'_0}(\mathbf{c_d}\varepsilon^d)^{+1}\\
&\qquad\leq p!\big(\mathbf{c_d}\varepsilon^d\big)^{p+1}nm\,e^{n+m+p}.
\end{aligned}\tag{4.15}$$

We can integrate on the rest of parameters using (4.3) and (4.4), and finally

$$\begin{aligned}
&\mu\left(\mathbb{E}_\varepsilon\left[\frac{1}{\mu^{n+m}}\sum_{\underline{i}_{n+m}}g_n(\mathbf{Z}_{\underline{i}_n})h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}})\right]-\mathbb{E}_\varepsilon\left[g_n\right]\mathbb{E}_\varepsilon\left[g\right]\right).\\
&\leq c_0c'_0\mu\sum_{p\geq 0}\frac{\mu^p}{p!}p!\big(\mathbf{c_d}\varepsilon^d\big)^{p+1}nm\,e^{n+m+p}\\
&\leq\mu\varepsilon^dnm(\mathbf{c_d}e)^{n+m}c_0c'_0\sum_{p\geq 0}(e\mathbf{c_d}\varepsilon)^p\\
&\leq(\varepsilon/\mathfrak{d})C^{n+m+1}\sum_{p\geq 0}(e\mathbf{c_d}\varepsilon/\mathfrak{d})^p
\end{aligned}$$

which converges for ε small enough.

- To prove (4.8), we apply the estimation (4.6) to (4.7):

$$\begin{aligned}\left|\mathbb{E}_\varepsilon\left[\mu\hat{g}_n\hat{h}_m\right]\right| &\leq \sum_{l=1}^{m}\binom{n}{l}\binom{m}{l}\frac{l!}{\mu^{l-1}}\frac{\mu^{l-1}C^l}{n^l}c_l + C^{n+m}c_0c_0'\frac{\varepsilon}{\mathfrak{d}}\\ &\leq \sum_{l=1}^{m}\binom{m}{l}\frac{n!}{(n-l)!n^l}C^lc_l + C^{n+m}c_0c_0'\\ &\leq (1+C)^m\max_{1\leq l\leq m}c_l + C^{n+m}c_0c_0'.\end{aligned}$$

□

Note also the following bound in L^p norms of the fluctuation field.

Theorem 8. *For any $p\in[2,\infty)$, there exists a constant $C_p>0$ such that*

$$\left(\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^p\right]\right)^{1/p} \leq C_p\|g\|_{L^p(M(v)dz)}. \tag{4.16}$$

The proof can be found in Appendix A of [BGSRS21].

From these estimations, one can deduce the following corollary:

Corollary 4.1. *Let h_n be a test function satisfying the conditions of Theorem 7. Then there exists a constant $C>0$ such that*

$$\begin{aligned}&\left|\mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_n}h_n(\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right|\\ &\qquad\leq C^n\mu^{n-1}\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{1/2}\left(c_0+\left(\sup_{1\leq l\leq n}c_l\right)^{1/2}\right).\end{aligned}\tag{4.17}$$

Proof.

$$\begin{aligned}\mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_n}h_n(\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right] &= \mu^{n-1}\mathbb{E}_\varepsilon\left[\mu^{\frac12-n}\sum_{\underline{i}_n}h_n(\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\\ &= \mu^{n-1}\left(\mathbb{E}_\varepsilon\left[\mu^{\frac12}\widehat{h_n}(\mathbf{Z}_{\mathcal{N}}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]+\mathbb{E}_\varepsilon\left[h_n\right]\mathbb{E}_\varepsilon\left[\mu^{\frac12}\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right)\\ &= \mu^{n-1}\left(\mathbb{E}_\varepsilon\left[\mu^{\frac12}\widehat{h_n}(\mathbf{Z}_{\mathcal{N}}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]+\mathbb{E}_\varepsilon\left[h_n\right]\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)\mu^{\frac12}\left(1-\mathbb{1}_{\Upsilon^c_\varepsilon}\right)\right]\right).\end{aligned}$$

By $\mathbb{E}_\varepsilon[\zeta^0_\varepsilon(g)]=0$ and using Cauchy-Schwarz inequality, we find

$$\begin{aligned}&\left|\mathbb{E}_\varepsilon\left[\mu^{-\frac12}\sum_{\underline{i}_n}h_n(\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right|\\ &\qquad\leq \mu^{n-1}\left(\mathbb{E}_\varepsilon\left[\mu\left[\widehat{h_n}\right]^2\right]^{\frac12}\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac12}+\mathbb{E}_\varepsilon\left[h_n\right]\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac12}\left(\mu\mathbb{P}_\varepsilon\left[\Upsilon^c_\varepsilon\right]\right)^{\frac12}\right).\end{aligned}$$

We apply now Theorem 7. The bound on $\mathbb{P}_\varepsilon\left[\Upsilon^c_\varepsilon\right]$ given in Section 3.2 and the bound on the L^p norm of $\zeta^0_\varepsilon(g)$ (4.16) lead to the stated corollary. □

5. Clustering estimations without recollision

The objective of this section is to bound $G^{\rm clust}_\varepsilon(t)$ and $G^{\rm exp}_\varepsilon(t)$, defined by

$$G^{\rm clust}_\varepsilon(t) := \mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right]-\sum_{\substack{(n_j)_{j\leq K}\\ 0\leq n_j-n_{j-1}\leq 2^j}}\mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_K}}\Psi^{0,t}_{\underline{n}_K}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right],$$

$$G^{\rm exp}_\varepsilon(t) := \sum_{1\leq k\leq K}\sum_{\substack{(n_j)_{j\leq k-1}\\ 0\leq n_j-n_{j-1}\leq 2^j}}\sum_{n_k\geq 2^k+n_{k-1}}\mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_k}}\Psi^{0,k\theta}_{\underline{n}_k}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right].$$

Proposition 5.1. *For $\varepsilon > 0$ small enough,*

$$|G_\varepsilon^{exp}(t) + G_\varepsilon^{rec}(t)| \le C\|g\|_0\|h\|_0 \left(\varepsilon^{1/3}(Ct)^{2^{t/\theta}} + \ t\theta^{1/2}\right) \tag{5.1}$$

To obtain the stated result, we need first the following bounds on the pseudotrajectory developments without recollisions of type $\Psi^{0,k\theta}_{\underline{n}_k}[h]$:

Proposition 5.2. *Fix $k \in \mathbb{N}$, $\underline{n} := (n_1, \cdots, n_k) \in \mathbb{N}^k$ with $n_1 \le n_2 \le \cdots \le n_k$ and define Then fixing $x_1 = 0$*

$$\int \sup_{y\in\mathbb{T}} \left|\Psi^{0,k\theta}_{\underline{n}_k}[h](\mathrm{tr}_y\, Z_{n_k})\right| \frac{e^{-\mathscr{H}_{n_k}(Z_{n_k})}}{(2\pi)^{\frac{n_k d}{2}}}\, \mathrm{d}V_{n_k}\, \mathrm{d}X_{2,n_k} \le \frac{\|h\|_0}{(\mu\mathfrak{d})^{n_k-1}} C^{n_k}\theta^{n_k-n_{k-1}}(k\theta)^{n_{k-1}-1}, \tag{5.2}$$

and, for $m \in [1, n_k]$,

$$\begin{aligned}&\int \sup_{y\in\mathbb{T}} \left|\Psi^{0,k\theta}_{\underline{n}_k}[h] \circledast_m \Psi^{0,k\theta}_{\underline{n}_k}[h](\mathrm{tr}_y\, Z_{2n_k-m})\right| \frac{e^{-\mathscr{H}_{2n_k-m}(Z_{2n_k-m})}}{(2\pi)^{\frac{(2n_k-m)d}{2}}}\, \mathrm{d}V_{2n_K-m}\, \mathrm{d}X_{2,2n_K-m} \\ &\qquad \le \frac{\mu^{m-1}}{n_k^m}\left(\frac{\|h\|_0}{(\mu\mathfrak{d})^{n_k-1}}C^{n_k}\right)^2 \theta^{n_k-n_{k-1}}(k\theta)^{n_{k-1}+n_k-1}.\end{aligned} \tag{5.3}$$

Using Corollary 4.1 and the previous estimations,

$$\begin{aligned}\left|\mathbb{E}_\varepsilon\left[\mu^{-\frac12}\sum_{\underline{i}_{n_k}} \Psi^{0,k\theta}_{\underline{n}_k}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right| &\le \|g\|_0\|h\|_0 C^{n_k}\Big((\tfrac{\theta}{\mathfrak{d}})^{n_k-n_{k-1}}(\tfrac{k\theta}{\mathfrak{d}})^{n_{k-1}-1} \\ &\qquad + (\tfrac{\theta}{\mathfrak{d}})^{\frac{n_k-n_{k-1}}{2}}(\tfrac{k\theta}{\mathfrak{d}})^{\frac{n_k+n_{k-1}-1}{2}}\Big) \\ &\le \|g\|_0\|h\|_0 C^{n_k}(\tfrac{\theta}{\mathfrak{d}})^{\frac{n_k-n_{k-1}}{2}}(\tfrac{t}{\mathfrak{d}})^{n_k},\end{aligned}$$

and in the same way,

$$\mathbb{E}_\varepsilon\left[\mu^{-\frac12}\sum_{\underline{i}_{n_K}} \Psi^{0,k\theta}_{\underline{n}_k}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right] = O\left(\varepsilon^{\frac12}\|g\|_0\|h\|_0 C^{n_k}(t/\mathfrak{d})^{n_k}\right).$$

Summing on all possible $(n_1, \cdots, n_k)$, we obtain

$$\begin{aligned}|G^{\exp}_\varepsilon(t)| &\le \sum_{k=1}^K \sum_{\substack{n_1\le\cdots\le n_{k-1}\\ n_j-n_{j-1}\le 2^j}} \sum_{n_k>2^k+n_{k-1}} \|g\|_0\|h\|_0 C^{n_k}(\theta/\mathfrak{d})^{\frac{n_k-n_{k-1}}{2}}(t/\mathfrak{d})^{n_k} \\ &\le \|g\|_0\|h\|_0 \sum_{k=1}^K \sum_{\substack{n_1\le\cdots\le n_{k-1}\\ n_j-n_{j-1}\le 2^j}} \sum_{n_k>2^k+n_{k-1}} \left(C\tfrac{\theta t^2}{\mathfrak{d}^3}\right)^{\frac{n_k-n_{k-1}}{2}} \\ &\le C\|g\|_0\|h\|_0 \sum_{k=1}^K 2^{k^2}\left(C\tfrac{\theta t^2}{\mathfrak{d}^3}\right)^{2^{k-1}} \le C\|g\|_0\|h\|_0 \tfrac{\theta t^2}{\mathfrak{d}^3}.\end{aligned}$$

as the series converges for θ small enough. In the same way

$$\begin{aligned}\left|G^{\mathrm{clust}}_\varepsilon(t)\right| \le& \mathbb{P}_\varepsilon(\Upsilon^c_\varepsilon)^{\frac14}\mathbb{E}_\varepsilon[\zeta^0_\varepsilon(g)^4]^{\frac14}\mathbb{E}_\varepsilon[\zeta^0_\varepsilon(h)^2]^{\frac12} \\ &+ \sum_{\substack{n_1\le\cdots\le n_K\\ n_j-n_{j-1}\le 2^j}} \mathbb{P}_\varepsilon(\Upsilon^c_\varepsilon)^{\frac14}\|g\|_0\|h\|_0 C^{n_k}t^{n_k} + \|g\|_0\|h\|_0(\tfrac{C\theta}{\mathfrak{d}})^{n_K}(\tfrac{\varepsilon}{\mathfrak{d}})^{\frac12} \\ \le& C\|g\|_0\|h\|_0\varepsilon^{\frac13}2^{K^2}(Ct)^{2^K}.\end{aligned}$$

This concludes the proof of (5.1).

Proof of (5.2). We recall that for $t = k\theta$ and that

$$\Psi^{0,k\theta}_{\underline{n}_k}[h] := \frac{1}{(n_k-1)!}\sum_{(s_l)_{l\le n_k-1}} \prod_{l=1}^{n_k-1} s_l h(\mathsf{z}_1(t,\cdot,\{1\},(s_l)_l))\mathbb{1}_{\mathscr{R}^{0,t}_{\{q\},(s_l)_l}} \prod_{i=0}^{k-1}\mathbb{1}_{\mathfrak{n}(i\theta)=n_{n_k-i}}.$$

This gives directly the following bound on $\bar{\Psi}^{0,t}_{\underline{n}_k}[h]$

$$\left|\bar{\Psi}^{0,t}_{\underline{n}_k}[h]\right| \leq \frac{\|h\|_0}{(n_k-1)!} \sum_{(s_l)_{l\leq n_k-1}} \sum_{(s_l)_{l\leq n_k-1}} \mathbb{1}_{\mathscr{R}^{0,t}_{\{1\},(s_l)_l}} \mathbb{1}_{\mathfrak{n}(\theta)=n_{k-1}}. \tag{5.4}$$

As the left-hand-side of (5.4) is invariant under translations, it is sufficient to fix $x_1 = 0$ and integrate with respect to (X_{2,n_k}, V_{n_k}).

We define the *clustering tree* $T^>$ as the sequence $(q_i, \bar{q}_i)_{1\leq i\leq n_k-1}$ where the i-th collision involves between particles q_i and $\bar{q}_i$ (and $q_i < \bar{q}_i$).

Since in the present Section, pseudotrajectories have no recollision, the clustering tree is just the collision graph where we forget the collisions times (but not their order). It can be used to parametrize a partition of $\mathscr{R}^{0,t}_{\{q\},(s_l)_l}$.

Let us fix a clustering tree. We perform the following change of variables

$$X_{2,n_k} \mapsto (\hat{x}_1, \cdots, \hat{x}_{n_k-1}),\ \forall i \in [1, n_k-1],\ \hat{x}_i := x_{q_i} - x_{\bar{q}_i}$$

Fix then τ_{i+1} the time of the $(i+1)$-th collision, as well as the relative positions $\hat{x}_1, \cdots, \hat{x}_{i-1}$. We denote $T_i = \theta$ if $i \leq n_k - n_{k-1}$, t else (at least $n_k - n_{k-1}$ clustering collisions happen before time θ). The i-th collision set is defined by

$$B_{T^>,i} := \left\{\hat{x}_i \middle| \exists \tau \in (0, T_i \wedge \tau_{i+1}),\ |\mathsf{x}_{q_i}(\tau) - \mathsf{x}_{\bar{q}_i}(\tau)| \leq \varepsilon\right\}.$$

Because particles $\mathsf{x}_{q_i}(\tau)$ and $\mathsf{x}_{\bar{q}_i}(\tau)$ are independent until their first meeting, we can perform the change of variable $\hat{x}_i \mapsto (\tau_i, \eta_i)$ where τ_i is the first meeting time and

$$\eta_i := \frac{\mathsf{x}_{q_i}(\tau_i) - \mathsf{x}_{\bar{q}_i}(\tau_i)}{|\mathsf{x}_{q_i}(\tau_i) - \mathsf{x}_{\bar{q}_i}(\tau_i)|}.$$

It sends the Lebesgue measure $\mathrm{d}\hat{x}_i$ to the measure $\varepsilon^{d-1}((\mathsf{v}_{q_i}(\tau_i) - \mathsf{v}_{\bar{q}_i}(\tau_i)) \cdot \eta_i)_+ \,\mathrm{d}\eta_i\, \mathrm{d}\tau_i$ and

$$\int \mathbb{1}_{B_{T^>,i}}\, \mathrm{d}\hat{x}_i \leq C\varepsilon^{d-1} |\mathsf{v}_{q_i}(\tau_i) - \mathsf{v}_{\bar{q}_i}(\tau_i)| \int_0^{T_i\wedge\tau_{i+1}} \mathrm{d}\tau_i.$$

We want sum now on every possible edge $(q_i, \bar{q}_i)$, hence, we need to control

$$\sum_{(q_i,\bar{q}_i)} |\mathsf{v}_{q_i}(\tau_i) - \mathsf{v}_{\bar{q}_i}(\tau_i)| \leq 2n_k \sum_k |\mathsf{v}_k(\tau_i)| \leq 2n_k \left(n_k \sum_k |\mathsf{v}_k(\tau_i)|^2\right)^{1/2} \leq n_k \left(n_k + |\mathsf{V}_{n_k}(\tau_i)|^2\right)$$

Lemma 5.3. *Consider a time $\tau \in [0,t]$, collision parameters $(\omega_1, \omega_2, (s_i)_i)$ and an initial position $Z_n \in \mathbb{D}^n$. Then*

$$\tfrac{1}{2}|\mathrm{V}(\tau, Z_n, \omega_1, \omega_2, (s_i)_i)|^2 \leq \mathscr{H}_n(Z_n)$$

as there is no overlap between particles.

Proof.

Definition 5.3.1. *Consider two times $0 \leq \tau_a < \tau_b \leq t$. We denote $\mathscr{G}$ the collision graph of the pseudotrajectory $\mathsf{Z}_{n_k}(\cdot, (\omega, (s_i)_i), Z_{n_k})$ on the time interval $[\tau_a, \tau_b]$ and G a graph with edges*

$$\left\{(q, \bar{q}) \in [n]^2,\ \exists \tau' \in [\tau_a, \tau_b],\ (q, \bar{q})_{1,\tau} \in \mathscr{G}\right\}.$$

We take only into account the collisions with interaction. We define $\underline{\kappa} := (\kappa_1, \cdots, \kappa_k)$the clusters on the segment $[\tau_1, \tau_2]$ *the connected components of G.*

Note that if τ_a lies between the beginning of the collision implying s_j and the beginning of the collision implying s_{j+1}, then $\underline{\kappa}$ only depends on the $(s_i)_{i\leq j}$.

We distinguished the cases $\tau > \delta$ and $\tau \leq \delta$.

- First, if $\tau \leq \delta$. We consider $\underline{(\kappa_1, \cdots, \kappa_k)}$ the cluster on the segment $[0, \delta]$. The pseudotrajectory is the Hamiltonian trajectory associated with the energy

$$\mathscr{H}_{\underline{\kappa}}(Z_n) := \sum_{i=1}^k \left(\sum_{q\in\kappa_i} \frac{|v_q|^2}{2} + \sum_{\substack{q,\bar{q}\in\kappa_i\\ q\neq\bar{q}}} \frac{\alpha}{2}\mathscr{V}\left(\frac{x_q - x_{\bar{q}}}{\varepsilon}\right)\right).$$

  Hence

$$\tfrac{1}{2}|\mathrm{V}(\tau, Z_n, \omega_1, \omega_2, (s_i)_i)|^2 \leq \mathscr{H}^{\underline{\kappa}}((\tau) \leq \mathscr{H}^{\underline{\kappa}}(Z_n) \leq \mathscr{H}_n(Z_n).$$

- If $\tau > \delta$, consider $\underline{\kappa}$ and $\underline{\kappa}'$ the clusters on $[\delta, \tau]$ and on $[0, \delta]$. After time δ the particles outside ω_2 stop interacting, and before time δ the couple of particles in ω_2 cannot overlap. Hence, $\underline{\kappa}'$ is a finer partition of $[n]$ than $\underline{\kappa}$ and $\mathscr{H}_{\underline{\kappa}} \leq \mathscr{H}_{\underline{\kappa}'}$. Thus

$$\frac{1}{2}|\mathsf{V}_{n_k}(\tau)|^2 \leq \mathscr{H}_{\underline{\kappa}}(\mathsf{Z}_{n_k}(\tau)) = \mathscr{H}_{\underline{\kappa}}(\mathsf{Z}_{n_k}(\delta)) \leq \mathscr{H}_{\underline{\kappa}'}(\mathsf{Z}_{n_k}(\delta)) = \mathscr{H}_{\underline{\kappa}'}(\mathsf{Z}_{n_k}(0)) \leq \mathscr{H}_n(Z_n). \tag{5.5}$$

□

Hence, using the Boltzmann-Grad scaling $\mu\varepsilon^{d-1}\mathfrak{d} = 1$,

$$\begin{aligned}
&\sum_{(q_i,\bar{q}_i)_i} \int \mathrm{d}\hat{x}_1 \mathbb{1}_{B_{T>,1}} \cdots \int \mathrm{d}\hat{x}_{n_k-1} \mathbb{1}_{B_{T>,n_k-1}} \mathbb{1}_{\mathfrak{n}(\theta)=n_{k-1}} e^{-\mathscr{H}_{n_k}(Z_{n_k})} \\
&\quad \leq \left(\frac{Cn_k}{\mu\mathfrak{d}}\right)^{n_k-1} (n_k + \mathscr{H}_{n_k}(Z_{n_k}))^{n_k-1}\, e^{-\mathscr{H}_{n_k}(Z_{n_k})} \int_0^{T_{n_k-1}} \mathrm{d}\tau_{n_k-1} \cdots \int_0^{T_1\wedge\tau_2} \mathrm{d}\tau_1 \\
&\quad \leq \left(\frac{Cn_k}{\mu\mathfrak{d}}\right)^{n_k-1} n_k^{n_k-1} e^{-\frac{\mathscr{H}_{n_k}(Z_{n_k})}{2}} \frac{t^{n_{k-1}-1}}{(n_{k-1}-1)!} \frac{\theta^{n_k-n_{k-1}}}{(n_k-n_{k-1})!} \\
&\quad \leq \left(\frac{\tilde{C}}{\mu\mathfrak{d}}\right)^{n_k-1} n_k^{n_k-1} e^{-\frac{|V_{n_k}|^2}{4}} t^{n_{k-1}-1}\theta^{n_k-n_{k-1}},
\end{aligned}$$

We used these to classical inequalities

$$\frac{(a+b)^{a+b}}{a!b!} \leq e^{a+b}\frac{(a+b)!}{a!b!} \leq (2e)^{a+b}$$

$$\forall A, B > 0,\ , x \in \mathbb{R}^+,\quad (A+x)^B\, e^{-\frac{x}{2}} = B^B \left(\frac{A+x}{B} e^{-\frac{A+x}{2B}}\right)^B e^{\frac{A}{4}} \leq \left(\tfrac{4B}{e}\right)^B e^{\frac{A}{4}}.$$

Finally, we sum on V_{n_k}, on the 2^{n_k-1} possible $(s_i)_i$ and on the $q \in [1, n_k]$, and we divide by the remaining $(n_k!)$. This gives the expected estimation. □

Proof of (5.3). We begin as in the previous paragraph

$$\begin{aligned}
&\left|\Psi^{0,t}_{\underline{n}_k}[h] \circledast_m \bar{\Psi}^{0,t}_{\underline{n}_k}[h](Z_{2n_k-m})\right| \\
&\quad \leq \frac{\|h\|_0^2}{(n_k!)^2} \frac{(n_k-m)!^2 m!}{(2n_k-m)!} \sum_{\substack{\omega\cup\omega'=[2n_k-m]\\ |\omega|=|\omega'|=n_k}} \sum_{\substack{q\in\omega,\\ q'\in\omega'}} \sum_{\substack{(s_l)_{l\leq n_k-1}\\ (s'_l)_{l\leq n_k-1}}} \mathbb{1}_{\mathscr{R}^{0,t}_{\{q\},(s_l)_l}}(Z_\omega)\mathbb{1}_{\mathfrak{n}(\theta)=n_{k-1}}(Z_\omega) \\
&\qquad\qquad \times \mathbb{1}_{\mathscr{R}^{0,t}_{\{q'\},(s'_l)_l}}(Z_{\omega'}).
\end{aligned} \tag{5.6}$$

where $\mathfrak{n}(\theta)$ is the number of particles at time θ in the pseudotrajectory $\mathsf{Z}(t, \cdot, \{1\}, (s_l)_l)$. The right hand-side is invariant under translation. Hence, without loss of generality we can suppose that $1 \notin \omega'$ and fix $x_1 = 0$.

We have to consider two pseudotrajectories

$$\mathsf{Z}(\tau) := \mathsf{Z}(\tau, Z_\omega, q, (s_l)_l) \text{ and } \mathsf{Z}'(\tau) := \mathsf{Z}(\tau, Z_{\omega'}, q', (s'_l)_l).$$

We want to estimates

$$\int \mathbb{1}_{\mathscr{R}^{0,t}_{\{q'\},(s'_l)_l}}(Z_{\omega'}) e^{-\frac{1}{2}\mathscr{H}_{2n_k-m}(Z_{2n_k-m})}\, \mathrm{d}Z_{\omega'\setminus\omega}.$$

Fix Z_ω and denote T_a the clustering tree of the pseudotrajectory $\mathsf{Z}(t)$, constructed as in the proof of (5.2). Next, we construct the clustering tree associated to the second pseudotrajectory: let $(q_i, \bar{q}_i)_{i\leq\ell}$ be the edges of the collision graph of $\mathsf{Z}'(\tau)$, taking temporal order. Set $\bar{T}_0 = \emptyset$. Suppose that $\bar{T}_i$ is constructed. Then $\bar{T}_{i+1} := \bar{T}_i \cup \{(q_i, \bar{q}_i)\}$ if the graph $T_a \cup \bar{T}_i \cup \{(q_i, \bar{q}_i)\}$ has no cycle. Else $\bar{T}_{i+1} := \bar{T}_i$. At the ℓ-step we have construct an ordered graph $T_b := \bar{T}_\ell$ with $n_k - m$ edges.

The T_b define a partition of $\{Z_{\omega'\setminus\omega} \in \mathbb{D}^{n_k-m} | Z_{\omega'} \in \mathscr{R}^{0,t}_{\{q'\},(s'_l)_l}\}$.

The rest of the proof is almost identical to the proof of (5.2). Fix the clustering tree $T_b = (q_i, \bar{q}_i)_{1\leq i\leq n_k-m}$, and perform the following change of variables

$$X_{\omega'\setminus\omega} \mapsto (\hat{x}_1, \cdots, \hat{x}_{2n_k-m-1}),\ \forall i \in [1, 2n_k-m-1],\ \hat{x}_i := x_{q_i} - x_{\bar{q}_i}.$$

Fix τ_{i+1}, the time of the $(i+1)$-th collision and relative positions $\hat{x}_1, \cdots, \hat{x}_{i-1}$. We define the i-th collision set as

$$B_{T>,i} := \left\{\hat{x}_i \middle| \exists\tau \in (0, T_i \wedge \tau_{i+1}),\ |\mathsf{x}'_{q_i}(\tau) - \mathsf{x}'_{\bar{q}_i}(\tau)| \leq \varepsilon\right\}$$

where $T_i = \theta$ for the $(n_k - n_{k-1})$ first collisions, t else.

As in the preceding lemma, we can perform the change of variable $\hat{x}_i \mapsto (\tau_i, \eta_i)$ where τ_i is the first meeting time and

$$\eta_i := \frac{\mathsf{x}'_{q_i}(\tau_i) - \mathsf{x}'_{\bar{q}_i}(\tau_i)}{\varepsilon} \text{ if the collision is in } \mathsf{Z}'(\cdot).$$

We have

$$\sum_{(q_i,\bar{q}_i)} \int \mathbb{1}_{B_{T>,i}} \,\mathrm{d}\hat{x}_i \leq C\varepsilon^{d-1} \sum_{(q_i,\bar{q}_i)\in\omega'^2} \left|\mathsf{v}'_{q_i}(\tau_i) - \mathsf{v}'_{\bar{q}_i}(\tau_i)\right| \int_0^{\tau_{i+1}} \mathrm{d}\tau_i.$$

Using the same method than in the proof of (5.5), we have

$$\begin{aligned}\sum_{(q_i,\bar{q}_i)\in\omega'^2} \left|\mathsf{v}'_{q_i}(\tau_i) - \mathsf{v}'_{\bar{q}_i}(\tau_i)\right| &\leq n_k + |\mathsf{V}'_{\omega'}(\tau_i)|^2 \leq 2n_k + 2\mathscr{H}_{n_k}(Z_{\omega'}) \\ &\leq 2n_k + 2\mathscr{H}_{2n_k-m}(Z_{2n_k-m}).\end{aligned}$$

By the same computation than above,

$$\begin{aligned}&\int \mathbb{1}_{\mathscr{R}^{0,t}_{\{q'\},(s'_l)_l}}(Z_{\omega'}) e^{-\frac{1}{2}\mathscr{H}_{2n_k-m}} \,\mathrm{d}Z_{\omega'\setminus\omega} \\ &\leq \sum_{T_b} \int e^{-\frac{\mathscr{H}_{2n_k-m}}{4}} \prod_{i=1}^{n_k-m} \mathbb{1}_{B_{T>,i}} \,\mathrm{d}\hat{x}_i \; e^{-\frac{|V_{\omega'}|^2}{8}} dV_{\omega'} \\ &\leq C\left(\frac{C}{\mu\mathfrak{d}}\right)^{n_k-m} t^{n_k-m}(2n_k-m)^{n_k-m}.\end{aligned}$$

We can estimate

$$\int \mathbb{1}_{\mathscr{R}^{0,t}_{\{q\},(s_l)_l}}(Z_\omega) e^{-\frac{1}{2}\mathscr{H}_{2n_k-m}} \,\mathrm{d}X_{\omega\setminus\{1\}} \,\mathrm{d}V_\omega$$

as in the proof of (5.2). We get the expected result by summing on all the possible parameters $(s_i)_i$, $(s'_i)_i$, q, q', ω and ω'. □

6. Treatment of the main part

The aim of this section the proof of

$$G^{\mathrm{main}}_\varepsilon(t) = \int_{\mathbb{D}} h(z)\mathbf{g}_\alpha(t,z)M(z)dz + O\left(\left(C\tfrac{\theta t}{\mathfrak{d}^2} + \varepsilon^{\mathfrak{a}} K 2^{K^2} (\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\right)\|h\|_1\|g\|_1\right),$$

where $\mathbf{g}_\alpha(t,z)$ is the solution of the Linearized Boltzmann equation (2.16).

6.1. Duality formula.

We recall that

$$\begin{aligned}G^{\mathrm{main}}_\varepsilon(t) &= \sum_{\substack{(n_j)_{j\leq K}\\ 0\leq n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_K}} \Psi^{0,t}_{\underline{n}_K}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\right] \\ &= \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\mu^{n_K}\,\hat{\Psi}^{0,t}_{\underline{n}_K}[h]\,\hat{g}\right]\end{aligned}$$

where $\Psi^{0,t}_{\underline{n}_K}[h]$ is the development of $h(z_i(t))$ along pseudotrajectories with n_k remaining particles at time $t-k\theta$ and no recollision, overlap nor multiple interaction.

We denote

$$\begin{aligned}g^\varepsilon_n(Z_n) &:= \left(\sum_{k=1}^n g(z_k)\right)\frac{1}{\mathscr{Z}}\sum_{p\geq 0}\frac{\mu^p}{p!}\int \frac{e^{\mathscr{H}_n(Z_n)-\mathscr{H}_{n+p}(Z_n,\underline{Z}_p)}}{(2\pi)^{\frac{dp}{2}}}\,\mathrm{d}\underline{Z}_p \\ &= \left(\sum_{k=1}^n g(z_k)\right)\sum_{p\geq 0}\frac{\mu^p}{p!}\int \psi^n_p(X_n,\underline{X}_p)\,\mathrm{d}\underline{X}_p.\end{aligned} \tag{6.1}$$

where the ψ^n_p are defined in (4.10).

Then using the equality (4.7) and L^1 estimations on $\Psi^{0,t}_{\underline{n}_K}[h]$ of Section 5, we have for h and g in L^∞

$$G^{\mathrm{main}}_\varepsilon(t) = \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} \mu^{n_K-1}\mathbb{E}_\varepsilon\left[\Psi^{0,t}_{\underline{n}_K,0}[h]\circledast_1 g\right] + O\left(\varepsilon\sum_{\underline{n}}(\tfrac{Ct}{\mathfrak{d}})^{n_k}\|h\|_0\|g\|_0\right)$$
$$= \sum_{\underline{n}_K}\int \mu^{n_K-1}\Psi^{0,t}_{\underline{n}_K}[h]\left(Z_{n_K}\right)g^\varepsilon_{n_K}(Z_{n_K})\frac{e^{-\mathscr{H}_{n_K}(Z_{n_K})}\,\mathrm{d}Z_{n_K}}{(2\pi)^{\frac{n_K d}{2}}} + O\left(\varepsilon\left(K2^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\|h\|_0\|g\|_0\right)\right).$$

We want to compute the asymptotics of each term in the sum. As we suppose there is no overlap

$$\int \mu_\varepsilon^{n_K-1}\Psi^{0,t}_{\underline{n}_K}[h]\left(Z_{n_K}\right)g^\varepsilon_{n_K}(Z_{n_K})\frac{e^{-\mathscr{H}_{n_K}(Z_{n_K})}\,\mathrm{d}Z_{n_K}}{(2\pi)^{\frac{n_K d}{2}}}$$
$$= \frac{\mu^{n_K-1}}{(n_K-1)!}\sum_{(s_k)_k}\prod_{k=1}^{n_l-1}s_k\int_{\mathscr{R}^{0,t}_{\{1\},(s_k)_k}} h(\mathsf{z}^\varepsilon_1(t,Z_{n_K},\{1\},(s_k)_k))g^\varepsilon_{n_K}(Z_{n_K})\prod_{i=1}^K \mathbb{1}_{\mathfrak{n}(t-i\theta)=n_i}M^{\otimes n_K}\,\mathrm{d}Z_{n_K}.$$

where $\mathscr{R}^{0,t}_{\{1\},(s_k)_k}$ is the set of initial parameters such that the pseudo trajectory has no recollision and $\mathfrak{n}(\tau)$ is the number of remaining particles at time τ (see definition 3.1.5). We had an exponent ε on z^ε_1 to mark the ε-dependence of the pseudotrajectory.

We want to construct the limiting process of the pseudotrajectory $\mathsf{Z}^\varepsilon_n(\tau)$.

We denote T the *clustering tree* as the sequence $(q_i,\bar{q}_i,\bar{s}_i)_{i\leq n_K-1}$ such that the i-th collision happens between particles q_i and $\bar{q}_i$ (with $q_i<\bar{q}_i$) and $\bar{s}_i$ is equal to 1 (respectively -1) if the particles interact (respectively do not interact). Fixing the initial velocities V_{n_K}, we perform the change of variable

$$X_{n_K}\mapsto(x_1,(\nu_i,\tau_i)_{i\leq n_K-1}),\ \text{with}\ \nu_i=\frac{\mathsf{x}^\varepsilon_{q_i}(\tau_i)-\mathsf{x}^\varepsilon_{\bar{q}_i}(\tau_i)}{\varepsilon}.$$

Its Jacobian is

$$\mathrm{d}X_{n_K}\to\prod_{i=1}^{n_k-1}\varepsilon^{d-1}\left(\left(\mathsf{v}^\varepsilon_{q_i}(\tau_i)-\mathsf{v}^\varepsilon_{\bar{q}_i}(\tau_i)\right)\cdot\nu_i\right)_+\,\mathrm{d}\nu_i\,\mathrm{d}\tau_i =: \frac{\Lambda_T(V_{n_K},\nu_{[n_k-1]})}{(\mu\mathfrak{d})^{n_k-1}}\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_K-1]}\ \ \mathrm{d}x_1,$$
$$\nu_{[n_k-1]}=(\nu_1,\cdots,\nu_{n_k-1}),\ \ \tau_{[n_k-1]}=(\tau_1,\cdots,\tau_{n_k-1}).$$

The kernel $\Lambda(V_{n_K},\nu_{[n_k-1]})$ only depends on the successive velocities $(\mathsf{v}^\varepsilon_{q_i}(\tau_i),\mathsf{v}^\varepsilon_{\bar{q}_i}(\tau_i))$ which can be deduced from the collision graph, forgetting the exact values of the $\mathrm{d}\tau_{[n_k-1]}$ (since we have forbid the pathological pseudotrajectories).

We defined the signature of the collision tree $\sigma(T):=\bar{s}_1\bar{s}_2\cdots\bar{s}_{n_K}$, the set of collision times

$$\mathfrak{T}_{\underline{n}_K}:=\{(\tau_i)_{i\leq n_K-1},\ \tau_i\leq\tau_{i+1},\ \forall k\leq K,\ j\in[n_K-n_{K-k},n_K-n_{K-k-1}],\ k\theta\leq\tau_j\leq(k+1)\theta\}$$

and for a given family $\tau_{[n_K-1]}$, we define $\mathfrak{G}^\varepsilon_T(\tau_{[n_K-1]})$ the set of coordinates $(x_1,(\nu_i)_{i\leq n_K-1},V_{n_K})$ such that the pseudotrajectory has no recollision and for all j, $\left(\mathsf{v}^\varepsilon_{q_i}(\tau_i)-\mathsf{v}^\varepsilon_{\bar{q}_i}(\tau_i)\right)\cdot\nu_i$ is positive. The map

$$\bigsqcup_{(s_k)_k}\{s_k\}\times\left(\mathscr{R}^{0,t}_{\{1\},(s_k)_k}\cap\{\text{no overlap}\}\cap\bigcap_{j\leq K-1}\{\mathfrak{n}(j\theta)=n_{K-j}\}\right)\to\bigsqcup_T\{T\}\times\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^\varepsilon_T$$
$$(X_{n_K},V_{n_K})\mapsto(x_1,(\nu_i,\tau_i)_{i\leq n_K-1},V_{n_K})$$

is a diffeomosphism and

$$\int \mu^{n_K-1}\Psi^{0,t}_{\underline{n}_K}[h]\left(Z_{n_K}\right)g^\varepsilon_{n_K}(Z_{n_K})^{\otimes n_K}\,\mathrm{d}Z_{n_K} \tag{6.2}$$
$$=\frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T\sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^\varepsilon_T} h(\mathsf{z}^\varepsilon_1(t,T)g^\varepsilon_{n_K}(\mathsf{Z}^\varepsilon_{n_K}(0,T))$$
$$\times M^{\otimes n_K}\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_K-1]}\,\mathrm{d}x_1\,\mathrm{d}V_{n_K}.$$

Definition 6.0.1 (Pseudotrajectories for punctual particles)**.** *Fix a collision tree $T:=(q_i,\bar{q}_i,\bar{s}_i)$ and collision parameters $(V_{n_k},\tau_{[n_k-1]},\nu_{[n_k-1]})$. We now define the pseudotrajectories for punctual particles. The velocities $\mathsf{V}^0_{n_K}(\tau,T)$ follow a jump process: at time 0, $\mathsf{V}^0_{n_K}(\tau=0,T)=V_{n_K}$. At time τ_i, if $\bar{s}_i=1$ the velocities of particles q_i, $\bar{q}_i$ jump to $v_{q_i}(\tau_i^+)$, $v_{\bar{q}_i}(\tau_i^+)$ given by $(v_{q_i}(\tau_i^+),v_{\bar{q}_i}(\tau_i^+),\tilde{\nu}_i):=\xi_\alpha(v_{q_i}(\tau_i^-),v_{\bar{q}_i}(\tau_i^-),\nu_i)$ (ξ_α the scattering map defined in)*

We defined $\mathfrak{G}^0_T$ *the set of the* $(x_1, (\nu_i)_{i\leq n_K-1}, V_{n_K})$ *such that for all* j, $\big(\mathsf{v}^0_{q_i}(\tau_i) - \mathsf{v}^0_{\bar q_i}(\tau_i)\big)\cdot\nu_i$ *is positive. Note that* $\mathfrak{G}^\varepsilon_T \subset \mathfrak{G}^0_T$.

Finally, we define the formal limit of the $g^\varepsilon_{n_K}$

$$g_{n_K}(Z_{n_K}) := \sum_{i=1}^{n_K} g(z_i).$$

We have formally the convergence

$$\begin{aligned}\int \mu^{n_K-1}\Psi^{0,t}_{\underline{n}_K}[h]\,(Z_{n_K})\, g^\varepsilon_{n_K}(Z_{n_K}) M^{\otimes n_K} dZ_{n_K} \\ \underset{\varepsilon\to 0}{\longrightarrow} \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^0_T} h(\mathsf{z}^0_1(t,T) g_{n_K}(\mathsf{Z}^0_{n_K}(0,T)) \\ \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\, \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K}.\end{aligned}$$

In order to have explicit rates of convergence, we decompose the error into three parts:

$$\begin{aligned}(6.3)\quad &\int \mu^{n_K-1}\,\Psi^{0,t}_{\underline{n}_K}[h] g^\varepsilon_{n_K} M^{\otimes n_K}\,\mathrm{d}Z_{n_K} \\ &= (6.4) + (6.5) + (6.6) + \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^0_T} h(\mathsf{z}^0_1(t,T) g_{n_K}(\mathsf{Z}^0_{n_K}(0,T)) \\ &\qquad \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\, \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K},\end{aligned}$$

where we define

$$\begin{aligned}(6.4)\quad &= \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^\varepsilon_T}\Big(h(\mathsf{z}^\varepsilon_1(t,T) g_{n_K}(\mathsf{Z}^\varepsilon_{n_K}(0,T)) - h(\mathsf{z}^0_1(t,T) g_{n_K}(\mathsf{Z}^0_{n_K}(0,T))\Big) \\ &\qquad \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\, \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K},\end{aligned}$$

$$\begin{aligned}(6.5)\quad &= -\frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{G}^0_T} h(\mathsf{z}^\varepsilon_1(t,T) g_{n_K}(\mathsf{Z}^\varepsilon_{n_K}(0,T))\left(1-\mathbb{1}_{\mathfrak{G}^\varepsilon_T}\right) \\ &\qquad \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\, \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K},\end{aligned}$$

$$\begin{aligned}(6.6)\quad &= \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^\varepsilon_T} h(\mathsf{z}^\varepsilon_1(t,T)\Big(g^\varepsilon_{n_K}(\mathsf{Z}^\varepsilon_{n_K}(0,T)) - g_{n_K}(\mathsf{Z}^\varepsilon_{n_K}(0,T))\Big) \\ &\qquad \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\, \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K}.\end{aligned}$$

The error parts are estimated using the following standard results:

Lemma 6.1. *Fix* $\bar n := (n_1,\cdots,n_k)$ *and denote for* $p\in[1,2]$

$$\Lambda^p_T(V_{n_K},\nu_{[n_k-1]}) := \prod_{i=1}^{n_k-1}\left|\mathsf{v}^0_{q_i}(\tau_i^-) - \mathsf{v}^0_{\bar q_i}(\tau_i^-)\right|^p$$

For any $\varepsilon>0$ *sufficiently small, we have*

$$(6.7)\quad \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T\sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^0_T}\Lambda^p(V_{n_K},(d\nu_i)_i)\,\mathrm{d}\tau_{[n_K-1]}\,\mathrm{d}x_1 M^{\otimes n_K}\,\mathrm{d}V_{n_K} \leq C^{n_K} t^{n_{K-1}-1}\theta^{n_K-n_{K-1}}.$$

Proof. Fix first the collision tree $T := (q_i,\bar q_i,\bar s_i)_i$. We sum on each ν_i in the decreasing order:

$$\begin{aligned}(6.8)\quad \sum_{(q_i,\bar q_i,\bar s_i)}\int\left|\mathsf{v}^0_{q_i}(\tau_i^-) - \mathsf{v}^0_{\bar q_i}(\tau_i^-)\right|^p\,\mathrm{d}\nu_i \leq C\sum_{(\bar q_i,\bar s_i)}\left|\mathsf{v}_{q_i}(\tau_i)-\mathsf{v}_{q_i}(\tau_i)\right|^p \leq Cn_K^{2-\frac p2}\left|\mathsf{V}_{n_K}(\tau_i)\right|^{\frac p2} \\ \leq Cn_K^{2-\frac p2}\left|V_{n_K}\right|^{\frac p2}\end{aligned}$$

using the conservation of energy.

Hence,

$$\sum_T \int \Lambda_T^p(V_{n_K}, \nu_{[n_k-1]})\, \mathrm{d}\nu_{[n_k-1]} M^{\otimes n_K} dV_{n_K} \leq C^{n_K} n_K^{2-\frac{p}{2}} \int |V_{n_K}|^{\frac{n_K p}{2}} M^{\otimes n_K}\, \mathrm{d}V_{n_K}$$
$$\leq C_1^{n_K} n_K^{2n_K} \int e^{-\frac{1}{4}|V_{n_K}|^2}\, \mathrm{d}V_{n_K} \leq C_2^{n_K} n_K^{2n_K}.$$

Integrating the collision times

$$\int_{\mathfrak{T}_{\underline{n}_K}} \mathrm{d}\tau_{[n_K-1]} \leq \prod_{k=0}^{K-1} \frac{\theta^{n_k - n_{k+1}}}{(n_k - n_{k+1})!} \leq \frac{((K-1)\theta)^{n_{K-1}-1}}{(n_{K-1}-1)!} \frac{\theta^{n_K - n_{K+1}}}{(n_K - n_{K+1})!}$$
$$\leq \frac{2^{n_K-1} t^{n_{K-1}-1} \theta^{n_K - n_{K+1}}}{(n_K-1)!}.$$

Finally, we multiply the two previous inequalities and $\frac{1}{(n_K-1)!}$. Using the Stirling formula, we obtain the expected estimation. □

Now we can bound (6.6). We recall that

$$g_n^\varepsilon(Z_n) = \left(\sum_{k=1}^n g(z_k)\right) \sum_{p\geq 0} \frac{\mu^p}{p!} \int \psi_p^n(X_n, \underline{X}_p)\, \mathrm{d}\underline{X}_p,$$

where the ψ_p^n are defined in (4.10). Using the estimation (4.13),

$$\sum_{p\geq 1} \frac{\mu_\varepsilon^p}{p!} \int \psi_p^{n_K}(X_{n_K}, \underline{X}_p)\, \mathrm{d}\underline{X}_p \leq \sum_{p\geq 1} \frac{\mu_\varepsilon^p}{p!} (p-1)! \big(Ce\varepsilon^d\big)^p n_K e^{n_K} \leq \sum_{p\geq 1} \big(C'\varepsilon\big)^p n_K e^{n_K} \leq 2\varepsilon n_K e^{n_K}$$

for ε small enough. Hence, $|g_n^\varepsilon(Z_n) - g_n(Z_n)|$ is smaller than $\|g\|_0 C^{n_K}\varepsilon$ and

$$|(6.6)| \leq \tilde{C}_1^{n_K} t^{n_K-1} \|g\|_0 \|h\|_0 \varepsilon.$$

Lemma 6.2. *Fix $\bar{n} := (n_1, \cdots, n_K)$. For any $\varepsilon > 0$ sufficiently small, we have*

$$\frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!} \sum_T \int_{\mathfrak{T}_{\underline{n}_K} \times \mathfrak{G}_T^0} \left|1 - \mathbb{1}_{\mathfrak{G}_T^\varepsilon}\right| \Lambda_T(V_{n_K}, \nu_{[n_K-1]})\, \mathrm{d}\nu_{[n_k-1]}\, \mathrm{d}\tau_{[n_K-1]}\ \ \mathrm{d}x_1 M^{\otimes n_K}\, \mathrm{d}V_{n_K}$$
$$\leq C^{n_K} t^{n_K+10} \varepsilon^{\mathfrak{a}}. \tag{6.9}$$

This is an estimation of the set of parameters leading to a pathology (a recollision, a triple interaction, or an overlap). It is proven in the Annex B.2. From Lemma 6.2 we deduce

$$|(6.5)| \leq C(Ct)^{n_K} \varepsilon^{\mathfrak{a}} \|g\|\, \|h\|.$$

Lemma 6.3. *Fix $\bar{n} := (n_1, \cdots, n_k)$, T, $\varepsilon > 0$ and $(x_1, (\tau_i, \nu_i)_i, V_{n_K}) \in \mathfrak{G}_T^\varepsilon$. We have*

$$\forall \tau \in [0,t],\ \left|\mathsf{X}_{n_K}^\varepsilon(\tau) - \mathsf{X}_{n_K}^0(\tau)\right| \leq \sum_{i=1}^{n_K-1} \frac{2n_K \mathbb{V}\varepsilon}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}. \tag{6.10}$$

Proof. Thanks to the estimation of the interaction time (B.1), the i-th collision lasts at most a time $\frac{\varepsilon}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}$. Hence, the two trajectories $\mathsf{X}_{n_K}^\varepsilon(\tau)$ and $\mathsf{X}_{n_K}^0(\tau)$ have coincident velocities for τ outside the union of the interval

$$\bigcup_{i=1}^{n_K-1} \left[\tau_i, \tau_i + \frac{\varepsilon}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}\right].$$

During a collision a particle can cross a distance smaller than $\frac{\varepsilon\mathbb{V}}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}$ which bound the error that a collision creates. Hence, after n_K collisions, summing on all the possible particles we obtain the expected bound.

□

Lemma 6.4. *Fix $\bar{n} := (n_1, \cdots, n_K)$. For any $\varepsilon > 0$ sufficiently small , we have*

$$|(6.4)| \leq C(Ct)^{n_K-1} \mathbb{V}\varepsilon \|g\|_1 \|h\|_1. \tag{6.11}$$

Proof. We have forbid any recollision, multiple interaction, and overlap. Hence, the velocities of pseudo-trajectories of particles of sizes ε and 0 coincide and

$$(6.4) \leq \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}_T^\varepsilon} \sum_{i=1}^{n_K-1} \frac{2n_K \mathbb{V}\varepsilon \|h\|_1\|g\|_1}{\left|\mathrm{v}_{q_i}(\tau_i^-)-\mathrm{v}_{q_i}(\tau_i^-)\right|} \times \Lambda(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_K-1]}\,\mathrm{d}x_1 M^{\otimes n_K}\,\mathrm{d}V_{n_K}$$
$$\leq \frac{\left(\frac{Ct}{\mathfrak{d}}\right)^{-n_K+1}\mathbb{V}\varepsilon}{(n_K!)^2}\|h\|_1\|g\|_1 \sum_{i=1}^{n_K-1}\sum_T\int_{\mathfrak{G}_T^\varepsilon} \frac{\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}M^{\otimes n_K}\,\mathrm{d}V_{n_K}}{\left|\mathrm{v}_{q_i}(\tau_i^-)-\mathrm{v}_{q_i}(\tau_i^-)\right|}.$$

We need to bound

$$\sum_T\int_{\mathfrak{G}_T^\varepsilon} \frac{\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}M^{\otimes n_K}\,\mathrm{d}V_{n_K}}{\left|\mathrm{v}_{q_i}(\tau_i^-)-\mathrm{v}_{q_i}(\tau_i^-)\right|}. \tag{6.12}$$

Note that $\mathrm{v}_q^0(\tau_i^+)$ does not depend on the $\tau_{[n_K-1]}$, but only on the order of the collisions.

Fix a collision tree $T=(q_i,q_i',s_i)_i$. We define for $i\in[1,n_K-1]$ the applications $\left(\Xi_T^i\right)_{1\leq i\leq n_K}$ as $\Xi_T^i=\mathrm{id}$ if $i=1$, and

$$\Xi_T^i : (V_{n_K},\nu_{[n_k-1]})\mapsto \begin{cases} \Big(v_1,\cdots,\overset{q_{i-1}}{v'_{q_{i-1}}},\cdots,\overset{q'_{i-1}}{v'_{q'_{i-1}}},\cdots,v_{n_K},\nu_1,\cdots,\overset{i-1}{\nu'_{i-1}},\cdots,\nu_{n_K-1}\Big) \text{ if } s_i=1\\ \Big(V_{n_k},\nu_1,\cdots,\overset{i-1}{-\nu_{i-1}},\cdots,\nu_{n_K-1}\Big) \text{ if } s_i=1\end{cases} \tag{6.13}$$

with the new velocities given by the scattering $(v'_{q_{i-1}},v'_{q'_{i-1}},\nu'_{i-1}):=\xi_\alpha((v'_{q_{i-1}},v'_{q'_{i-1}},\nu'_{i-1}))$. We have that

$$(\mathrm{V}_{n_K}(\tau_i^-),\nu^i_{[n_K-1]}):=\Xi_T^i\Xi_T^{i-1}\cdots\Xi_T^1(V_{n_K},\nu_{[n_K-1]}).$$

Using that the Jacobian of the scattering ξ_α is 1 and the conservation by the scattering of the energy and angular momentum, the Jacobian of the transformation $\Xi_T^i\Xi_T^{i-1}\cdots\Xi_T^1$ is

$$\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}x_1M^{\otimes n_K}\,\mathrm{d}V_{n_K}\to\Lambda_T^{(i)}(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}x_1M^{\otimes n_K}\,\mathrm{d}V_{n_K}$$

where we start now the velocity process at time τ_i^- with $\mathrm{V}_{n_K}(\tau_i^-):=V_{n_K}$ and

$$\Lambda_T^{(i)}(V_{n_K},\nu_{[n_k-1]}):=\prod_{j=1}^{i-1}\left(\left(\mathsf{v}_{q_j}(\tau_j^+)-\mathsf{v}_{\bar q_j}(\tau_j^+)\right)\cdot\nu_j\right)_+\prod_{j=i}^{n_k-1}\left(\left(\mathsf{v}_{q_j}(\tau_j^-)-\mathsf{v}_{\bar q_j}(\tau_j^-)\right)\cdot\nu_j\right)_-.$$

Hence,

$$(6.12)\leq\sum_T\int_{\mathfrak{G}_T^\varepsilon}\frac{\Lambda_T^{(i)}(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}M^{\otimes n_K}\,\mathrm{d}V_{n_K}}{|v_{q_i}-v_{q_i}|}$$

Using the usual bound on $\sum_T\Lambda(V_{n_K},\ \mathrm{d}\nu_{[n_k-1]})$ that can be adapted to Λ_i,

$$(6.12)\leq\sum_{(q,q')}\int\frac{(Cn_K|V_{n_K}|^2+1)^{n_K}M^{\otimes n_K}\,\mathrm{d}V_{n_K}}{|v_q-v_{q'}|}\leq n_K^2(C'n_K^2)^{n_K}\int\frac{e^{\frac{|V_{n_K}|^2}{4}}\,\mathrm{d}V_{n_K}}{|v_1-v_2|}\leq(C''n_K^2)^{n_K}$$

as $\frac{1}{|v_1-v_2|}$ is an integrable singularity. This conclude the proof. □

Finally we get for h and g Lipschitz

$$\int\mu_\varepsilon^{n_K-1}\,\Psi^{0,t}_{\underline{n}_K}[h]\,g^\varepsilon_{n_K}M^{\otimes n_K}\,\mathrm{d}Z_{n_K}=\frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T\sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}_T^0}h(\mathsf{z}_1^0(t,T)g_{n_K}(\mathsf{Z}^0_{n_K}(0,T))$$
$$\times\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_K-1]}\,\mathrm{d}x_1M^{\otimes n_K}\,\mathrm{d}V_{n_K}$$
$$+O\left(\varepsilon^{\mathfrak{a}}\left(\tfrac{Ct}{\mathfrak{d}}\right)^{n_K}\|h\|_1\|g\|_1\right).$$

and therefore

$$(6.14)\quad G_\varepsilon^{\mathrm{main}}(t) = \sum_{\substack{n_1\le\cdots\le n_K\\ n_j-n_{j-1}\le 2^j}} \frac{\mathfrak{d}^{-n_K+1}}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}_T^0} h(\mathsf{z}_1^0(t,T) g_{n_K}(\mathsf{Z}_{n_K}^0(0,T)) \times \Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_K-1]}\,\,\mathrm{d}x_1 M^{\otimes n_K}\,\mathrm{d}V_{n_K} + O\left(\varepsilon^{\mathfrak{a}} K 2^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\|h\|_1\|g\|_1\right).$$

6.2. **Linearized Boltzmann equation.** We now identify the main part of (6.14).

Let $\mathbf{g}_\alpha$ be the solution of the Linearized Boltzmann equation

$$\partial_t\mathbf{g}_\alpha(t)+v\cdot\nabla_x\mathbf{g}_\alpha(t)=\frac{1}{\mathfrak{d}}\mathscr{L}_\alpha\mathbf{g}_\alpha(t),\qquad \mathbf{g}_\alpha(t=0)=g$$

where $\mathscr{L}_\alpha$ is the linearized Boltzmann operator associated to the potential $\alpha\mathscr{V}(\cdot)$

$$\mathscr{L}_\alpha g(v):=\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big(g(v')+g(v'_*)-g(v)-g(v_*)\big)M(v_*)\big((v-v_*)\cdot\nu\big)_+\,\mathrm{d}\nu\,\,\mathrm{d}v_*.$$

This equation can be rewritten in the Duhamel form:

$$\mathbf{g}_\alpha(t)=S(t)g+\frac{1}{\mathfrak{d}}\int_0^t S(t-\tau_1)\mathscr{L}_\alpha\mathbf{g}_\alpha(\tau_1)\,\mathrm{d}\tau_1$$

where $S(\tau)$ is the free transport

$$S(\tau)g(x,v)=g(x-tv,v).$$

We iterate this formula, but we still want to cut the cases with too many collisions in a short time interval (as for the particle system). Let define

$$Q_{m,n}(\tau)[g]=\frac{1}{\mathfrak{d}^{m-n}}\int_0^\tau\mathrm{d}\tau_n\int_0^{\tau_n}\cdots\int_0^{t_{m+2}}\mathrm{d}\tau_{m+1}S(t-\tau_n)\mathscr{L}_\alpha S(\tau_n-\tau_{n-1})\cdots\mathscr{L}_\alpha S(\tau_{m+1})g,$$

and for $\underline{n}_k:=(n_1,\cdots,n_k)$ with $1\le n_1\le\cdots\le n_k$,

$$Q_{\underline{n}_k}(\tau)g=Q_{1,n_1}(\tfrac{\tau}{k})Q_{n_1,n_2}(\tfrac{\tau}{k})\cdots Q_{n_{k-1},n_k}(\tfrac{\tau}{k})[g].$$

We have

$$(6.15)\qquad \mathbf{g}(t)=\sum_{\substack{n_1\le\cdots\le n_K\\ n_j-n_{j-1}\le 2^j}}Q_{\underline{n}_k}(t)[g]+\sum_{k=1}^K\sum_{\substack{n_1\le\cdots\le n_{k-1}\\ n_j-n_{j-1}\le 2^j}}\sum_{n_k>+n_{k-1}+2^k}Q_{\underline{n}_k}(k\theta)[\mathbf{g}_\alpha(t-k\theta)].$$

In a first time we bound the term of the sum: we have the classical estimates

Proposition 6.5. *There exists a constant C such that for any $g\in L^2(M(v)dz)$, and $\underline{n}:=(n_1,\cdots,n_k)$,*

$$(6.16)\qquad \big\|Q_{\underline{n}}(k\theta)g\big\|_{L^2(M^2(v)dz)}\le\big(\tfrac{C(k-1)\theta}{\mathfrak{d}}\big)^{\frac{n_{k-1}}{2}}\big(\tfrac{C\theta}{\mathfrak{d}}\big)^{\frac{n_k-n_{k-1}}{2}}\|g\|_{L^2(M(v)dz)}.$$

The proof is the same than the one of Proposition 7.5 of [LB22].

Because $\mathbf{g}_\alpha(t)$ is bounded in $L^\infty_t L^2(M(v)dz)$ by $\|g\|_{L^2(M(v)dz)} \leq C\|g\|_0$, we can bound the rest term of (6.15) by

$$\begin{aligned}
&\left|\sum_{k=1}^{K} \sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}} \sum_{n_k>+n_{k-1}+2^k} \int h(z) Q_{\underline{n}_k}(k\theta)[\mathbf{g}_\alpha(t-k\theta)](z) M(v)\,\mathrm{d}z\right| \\
&\qquad\leq \sum_{k=1}^{K} \sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}} \sum_{n_k>+n_{k-1}+2^k} \left(\tfrac{C(k-1)\theta}{\mathfrak{d}}\right)^{\frac{n_{k-1}}{2}} \left(\tfrac{C\theta}{\mathfrak{d}}\right)^{\frac{n_k-n_{k-1}}{2}} \|g\|_0\|h\|_0 \\
&\qquad\leq \sum_{k=1}^{K} \sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}} \sum_{n_k>2^k+n_{k-1}} \left(C\tfrac{\theta t}{\mathfrak{d}^2}\right)^{\frac{n_k-n_{k-1}}{2}} \|g\|_0\|h\|_0 \\
&\qquad\leq C\sum_{k=1}^{K} 2^{k^2}\left(C\tfrac{\theta t^2}{\mathfrak{d}^2}\right)^{2^{k-1}} \|g\|\|h\| \leq C\tfrac{\theta t}{\mathfrak{d}^2}\|g\|_0\|h\|_0.
\end{aligned} \tag{6.17}$$

The series converges since $\frac{\theta t^2}{\mathfrak{d}^2} < 1$.

The final step is the identification of the main part in (6.15):

Proposition 6.6. *Fix $\underline{n}_K := (n_1,\cdots,n_K)$ an increasing sequence of integer. Then*

$$\begin{aligned}
\int_{\mathbb{D}} h(z) Q_{\underline{n}_k}(t)[g](z) M(v) dz = \frac{\mathfrak{d}^{-n_k+1}}{(n_k-1)!} \sum_T \sigma(T) \int_{\mathfrak{T}_{\underline{n}_K}\times\mathfrak{G}^0_T} h(\mathsf{z}^0_1(t,T) g_{n_k}(\mathsf{Z}^0_{n_k}(0,T)) \\
\times\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_k-1]}\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k}.
\end{aligned} \tag{6.18}$$

Proof. We fix for the moment the collision times $(\tau_i)_i$).

We begin by developing g_{n_K}:

$$\begin{aligned}
&\sum_T \sigma(T) \int_{\mathbb{G}^0_T} h(\mathsf{z}^0_1(t,T)) g_{n_k}(\mathsf{Z}^0_{n_k}(0,T)) \Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_k-1]}\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k} \\
&\quad= \sum_{q_f=1}^{n_k}\sum_T \sigma(T) \int_{\mathbb{G}^0_T} h(\mathsf{z}^0_1(t,T)) g(\mathsf{z}^0_{q_f}(0,T)) \Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_k-1]}\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k}.
\end{aligned} \tag{6.19}$$

Definition 6.6.1. *Fix a collision tree $T := (q_i,\bar{q}_i,\bar{s}_i)_{i\leq n_k-1)}$ and a particle q_f. We say that a sequence $(i_1,\cdots,i_\ell)$ is* causal *if*

$$i_1<\cdots<i_\ell,\ \forall j<\ell,\ \{q_{i_j},\bar{q}_{i_j}\}\cap\{q_{i_{j+1}},\bar{q}_{i_{j+1}}\}\neq\emptyset.$$

A particle $\bar{q}$ influences *the particle 1 (respectively q_f) if there exists a causal sequence $(i_1,\cdots,i_\ell)$ such that $\bar{q}\in\{q_{i_1},\bar{q}_{i_1}\}$ and $1\in\{q_{i_\ell},\bar{q}_{i_\ell}\}$ (respectively $q_f\in\{q_{i_1},\bar{q}_{i_1}\}$ and $\bar{q}\in\{q_{i_\ell},\bar{q}_{i_\ell}\}$).*

If there exists a particle $\bar{q}$ which has only one collision $\bar{\imath}$ and which does not influence both particles 1 and q_f.

We use now the application Ξ^i_T defined in (6.13). We recall that

$$\Xi^{\bar{\imath}}_T\Xi^{\bar{\imath}-1}_T\cdots\Xi^1_T(V_{n_K},\nu_{[n_K-1]}) = (\tilde{V}_{n_K} = \mathsf{V}_{n_K}(\tau^-_{\bar{\imath}}),\tilde{\nu}_{[n_K-1]}).$$

In a second time, for a fix $(V_{n_K},\nu_{[n_K-1]})$, we perform the translation $x_1\mapsto\tilde{x}_1 := \mathsf{x}_1(\tau_{\bar{\imath}})$. The Jacobian of $\Xi^{\bar{\imath}}_T\Xi^{\bar{\imath}-1}_T\cdots\Xi^1_T$ is

$$\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}x_1 M^{\otimes n_K}\,\mathrm{d}V_{n_K} \to \Lambda^{(i)}_T(\tilde{V}_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tilde{x}_1 M^{\otimes n_K}\,\mathrm{d}\tilde{V}_{n_K}.$$

We start now the velocity process at time τ^-_i with $\mathrm{V}_{n_K}(\tau^-_i) := \tilde{V}_{n_K}$ and

$$\Lambda^{(i)}_T(\tilde{V}_{n_K},\nu_{[n_k-1]}) := \prod_{j=1}^{i-1}\left(\left(\mathsf{v}_{q_j}(\tau^+_j)-\mathsf{v}_{\bar{q}_j}(\tau^+_j)\right)\cdot\nu_j\right)_+ \prod_{j=i}^{n_k-1}\left(\left(\mathsf{v}_{q_j}(\tau^-_j)-\mathsf{v}_{\bar{q}_j}(\tau^-_j)\right)\cdot\nu_j\right)_- .$$

We pair T with the tree $\tilde{T}$ as

$$\tilde{T} := \begin{cases} (q_j,\bar{q}_j,\bar{s}_j) & \text{for } j\neq i \\ (q_j,\bar{q}_j,-\bar{s}_j) & \text{for } j=i. \end{cases}$$

Then $\sigma(T) = -\sigma(\tilde{T})$, and for same $\tilde{V}_{n_K}, \tau_{[n_K-1]}, \nu_{[n_K-1]}$ we have $\mathsf{z}_1^0(t,T) = \mathsf{z}_1^0(t,\tilde{T})$ and $\mathsf{z}_q^0(0,T) = \mathsf{z}_q^0(0,\tilde{T})$. We have $\Lambda_T^{(i)}(V_{n_K}, \nu_{[n_k-1]}) = \Lambda_{\tilde{T}}^{(i)}(V_{n_K}, \nu_{[n_k-1]})$. Thus

$$\sigma(T)\int_{\mathbb{G}_T^0} h(\mathsf{z}_1^0(t,T))g(\mathsf{z}_q^0(0,T))\Lambda(V_{n_K},\,\mathrm{d}\nu_{[n_k-1]})\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k}$$
$$= -\sigma(\tilde{T})\int_{\mathbb{G}_{\tilde{T}}^0} h(\mathsf{z}_1^0(t,\tilde{T}))g(\mathsf{z}_q^0(0,\tilde{T}))\Lambda(V_{n_K},\,\mathrm{d}\nu_{[n_k-1]})\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k}.$$

Hence, it remains only in (6.19) the trees such that every particle influence both 1 and q_f. The other terms are exactly compensated.

For a remaining tree $T = (q_i, \bar{q}_i, \nu_i)_i$ we can construct the sequences

$$\tilde{q}_0 := q_f,\ \tilde{q}_{n_K-1} := 1,\ \{\tilde{q}_i\} := \{q_i,\bar{q}_i\}\cap\{q_{i+1},\bar{q}_{i+1}\},\ \{\tilde{q}_i'\} := \{q_i,\bar{q}_i\}\setminus\{\tilde{q}_i\}$$

and

$$\tilde{s}_i := \begin{cases} 1 \text{ if } q_i = q_{i-1} \\ -1 \text{ else.}\end{cases}$$

The sequence $(\tilde{q}_i')_i$ encodes the order in which particles collide. In addition, we can reconstruct T for a given sequence $(\bar{s}_i, \tilde{s}_i, \tilde{q}_i')_i$.

We can reorder the particles such that $\tilde{q}_i' = n_K - i$ (there are $(n_K-1)!$ possibility).

Finally, we have to identify the four possible $(\bar{s}_i, \tilde{s}_i)_i$ with the four parts of $\mathscr{L}_\alpha$: $(1,1)$ with $g(v')$ (we follow the same particle which is deviated by the collision), $(1,-1)$ with $-g(v)$ (we follow the same particle which is not deviated by the collision), $(-1,1)$ with $g(v_*')$ and $(-1,-1)$ with $-g(v_*)$. There are $(n_K-1)!$ possible sequence $(\tilde{q}_i')_i$.

We conclude that

$$\frac{1}{(n_K-1)!}\sum_T \sigma(T)\int_{\mathbb{G}_T^0} h(\mathsf{z}_1^0(t,T))g_{n_k}(\mathsf{Z}_{n_k}^0(0,T))\Lambda(V_{n_K},\,\mathrm{d}\nu_{[n_k-1]})\,\mathrm{d}x_1 M^{\otimes n_k}\,\mathrm{d}V_{n_k}$$
$$= \int_{\mathbb{D}} h(z)S(t-\tau_{n_k-1})\mathscr{L}_\alpha S(\tau_{n_k-1}-\tau_{n_k-2})\cdots\mathscr{L}_\alpha S(\tau_1)g(z)\ M(v)\,\mathrm{d}z.$$

We obtain the expected result by integrating with respect to $(\tau_1,\cdots,\tau_{n_k-1})$. □

Combining the preceding proposition and the estimations (6.17) and (6.14), we obtain:

$$G_\varepsilon^{\mathrm{main}}(t) = \int_{\mathbb{D}} h(z)\mathbf{g}_\alpha(t,z)M(z)\,\mathrm{d}z + O\left(\left(\tfrac{\theta t}{\mathfrak{d}^2} + \varepsilon^{\mathfrak{a}}K2^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^{K+1}}\right)\|h\|_1\|g\|_1\right). \tag{6.20}$$

7. Estimation of non-pathological recollisions

In the last two sections, we estimate the error terms where the pseudotrajectory can have a recollision. We begin with the case of non pathological recollision.

$$G_\varepsilon^{\mathrm{rec},1}(t) = \sum_{\substack{0\le k\le K-1\\ 1\le k'\le K'}}\ \sum_{\substack{(n_j)_{j\le k}\\ 0\le n_j-n_{j-1}\le 2^j}}\ \sum_{n_{k+2}\ge n_{k+2}\ge n_k}\mathbb{E}_\varepsilon\left[\frac{1}{\sqrt{\mu}}\sum_{\underline{i}_{n_k}}\Psi^{>,t-t_s}_{\underline{n}_{k+1}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t_s)\right)\zeta_\varepsilon^0(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]$$

where $t - t_s = k\theta + k'\theta'$.

Proposition 7.1. *For ε small enough,*

$$\left|G_\varepsilon^{rec,1}(t)\right| \le \|g\|\|h\|\varepsilon^{\mathfrak{a}/2}(C't)^{2^{t/\theta}+2d+6}. \tag{7.1}$$

It is sufficient to prove the two following estimations:

Proposition 7.2. *Fix $k\in\mathbb{N}$, $\underline{n} := (n_1,\cdots,n_{k+2})\in\mathbb{N}^k$. Then denoting*

$$\bar{\Psi}^{>,t-t_s}_{\underline{n}_{k+2}!}[h](Z_{n_{k+2}}) := \frac{1}{n_{k+2}}\sum_{\sigma\in\mathfrak{S}_{n_{k+2}}}\Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h](Z_{\sigma[n_{k+2}]}), \tag{7.2}$$

and fixing $x_1 = 0$ *we have*

$$\int \sup_{y\in\mathbb{T}} \left|\Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h](\mathrm{tr}_y\, Z_{n_{k+2}})\right| \frac{e^{-\mathscr{H}_{n_{k+2}}}}{(2\pi)^{\frac{n_k d}{d}}}\, \mathrm{d}V_{n_{k+2}}\, \mathrm{d}X_{2,n_{k+2}} \leq \varepsilon^{\mathfrak{a}} \frac{\|h\|_0}{(\mu\mathfrak{d})^{n_{k+2}-1}} C^{n_{k+2}} \delta^2 \theta^{(n_{k+2}-n_k-3)_+} t^{n_k+9+d} \varepsilon^{\mathfrak{a}}, \tag{7.3}$$

and, for $m \in [1, n_{k+2}]$,

$$\int \sup_{y\in\mathbb{T}} \left|\Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h] \circledast_l \Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h](\mathrm{tr}_y\, Z_{2n_{k+2}-m})\right| M^{\otimes(2n_{k+2}-m)}\, \mathrm{d}V_{2n_{n+2}-m}\, \mathrm{d}X_{2,2n_{k+2}-m} \leq \frac{\mu^{m-1}}{n_{k+2}^m} \varepsilon^{\mathfrak{a}} \left(\frac{\|h\|_0}{(\mu\mathfrak{d})^{n_{k+2}-1}} C^{n_{k+2}}\right)^2 \delta^2 \theta^{(n_{k+2}-n_k-3)_+} t^{n_k+n_{k+2}+9+d}. \tag{7.4}$$

Using these estimations and Corollary 4.1,

$$\begin{aligned}
&\left|\mathbb{E}_\varepsilon\left[\mu^{-\frac12} \sum_{\underline{i}_{n_{k+2}}} \Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h] \left(\mathbf{Z}_{\underline{i}_{n_{k+2}}}(t_s)\right) \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon}\right]\right| \\
&\quad \leq \|h\|_0 \|g\|_0 \frac{C^{n_{k+2}}}{\mathfrak{d}^{n_{k+2}-1}} \left(\varepsilon^{\frac12+\mathfrak{a}} \theta^{(n_{k+2}-n_k-2)_+} \delta^2 t^{n_k+d+9} + \left(e^{\mathfrak{a}} \theta^{(n_{k+2}-n_k-2)_+} \delta^2 t^{n_k+d+9+m}\right)^{\frac12}\right) \\
&\quad \leq \|g\|_0 \|h\|_0 \delta \varepsilon^{\frac{\mathfrak{a}}{2}} C^{n_{k+2}} \left(\tfrac{\theta}{\mathfrak{d}}\right)^{\frac{(n_{k+2}-n_k-2)_+}{2}} \left(\tfrac{t}{\mathfrak{d}}\right)^{\frac{n_{k+2}+n_k}{2}+d+9} \\
&\quad \leq \|g\|_0 \|h\|_0 \delta \varepsilon^{\frac{\mathfrak{a}}{2}} \left(\tfrac{Ct}{\mathfrak{d}}\right)^{n_k+d+9} \left(\tfrac{Ct\theta}{\mathfrak{d}^2}\right)^{\frac{(n_{k+2}-n_k-2)_+}{2}}.
\end{aligned}$$

Using that $\frac{Ct\theta}{\mathfrak{d}^2} \leq 1$ and $K'\delta = \theta \leq 1$, we can sum on k, k' and $\underline{n}_{k+2}$

$$\begin{aligned}
\left|G^{\mathrm{rec},1}_\varepsilon(t)\right| &\leq \sum_{\substack{1\leq k\leq K-1\\ 1\leq k'\leq K'}} \sum_{\substack{n_1\leq\cdots\leq n_k\\ n_j-n_{j-1}\leq 2^j}} \sum_{n_{k+2}\geq n_{k+1}\geq n_k} \|g\|_0 \|h\|_0 \delta \varepsilon^{\frac{\mathfrak{a}}{2}} \left(\tfrac{Ct}{\mathfrak{d}}\right)^{n_k+d+} \left(\tfrac{Ct\theta}{\mathfrak{d}^2}\right)^{\frac{(n_{k+2}-n_k-2)_+}{2}} \\
&\leq \|g\|_0 \|h\|_0 K'\delta \varepsilon^{\frac{\mathfrak{a}}{2}} K^{K^2} \left(\tfrac{Ct}{\mathfrak{d}}\right)^{2^K+d+9} \\
&\leq \|g\|_0 \|h\|_0 \varepsilon^{\frac{\mathfrak{a}}{2}} \left(\tfrac{C't}{2\mathfrak{d}}\right)^{2^K+d+9}
\end{aligned}$$

This conclude the proof of (7.1).

Proof of (7.3). We recall that the pseudotrajectory development takes the form

$$\Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h] := \frac{1}{(n_{k+2}-1)!} \sum_{\substack{1\in\omega\subset[n_{k+2}]\\ |\omega|=n_{k+1}\\ (s_i)_{i\leq n_{i+2}-1}}} \prod_{k=1}^{n_l-1} s_k\, h(\mathsf{z}_q(t,\cdot,\{q\},\omega,(s_i)_i)) \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}} \prod_{i=1}^{k} \mathbb{1}_{\mathfrak{n}(t-i\theta)=n_i}.$$

Here $\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}$ is the set of initial configurations $Z_{n_{k+2}}$ such that the pseudotrajectory has

- 1 the final particle at time $t - t_s$,
- ω particles at time δ,
- at least one recollision,
- with no pathological recollision (thanks to the asymmetric conditioning).

Lemma 7.3. *There exists a constant* $\mathfrak{a} \in (0,1)$ *such that for any* $\underline{n}$, k' *and* $(s_i)_i$,

$$\int \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}} M^{\otimes n_{k+2}}\, \mathrm{d}X_{2,n_{k+2}}\, \mathrm{d}V_{n_{k+2}} \leq C' \left(\frac{C'}{\mu\mathfrak{d}}\right)^{n_{k+2}-1} (n_{k+2})^{n_{k+2}} \delta^2 \theta^{(n_{k+2}-n_k-2)_+} t^{n_k+2d+4} \varepsilon^{\mathfrak{a}}. \tag{7.5}$$

Proof. We may define the clustering tree $T^>$ as before, by looking at collisions in temporal order and keeping only the clustering collisions. However, this will not be sufficient to characterise the initial data.

Let $(q, \bar{q})$ (with $q < \bar{q}$) be the first two particles having a non-clustering collision, τ_{cycle} the time of this collision, and $c \in [1, n_{k+2}-1]$ such that τ_{cycle} lies between the times of the c-th and the $(c+1)$-th clustering collision. The parameters $(T^>, (q, \bar{q}, c))$ provide a partition of the set of initial data.

We denote

$$\mathfrak{T}_{\underline{n}_{k+2}} := \left\{ \begin{array}{c} (\tau_i)_{i\le n_{k+2}-1},\ \tau_i \le \tau_{i+1}, \\ \forall j \le n_{k+2}-n_{k+1},\ \tau_j\le\delta \\ \forall j \le n_{k+2}-n_k[,\ \tau_j \le k'\delta \\ \forall \ell\le k,\ j\le n_{k+2}-n_{k+2-\ell-1},\ \tau_j\le k'\delta+(\ell+1)\theta \end{array}\right\}$$

For a given initial data Z_n we denote $T := (q_i, q_i', s_i)_i$ the clustering tree, τ_i defined as the time of the i-th clustering collision and $\nu_i := (\mathrm{x}_{q_i}(\tau_i) - \mathrm{x}_{q_i}(\tau_i'))/\varepsilon$. We denote $\mathfrak{T}_{\underline{n}_{k+2}} \times \mathfrak{G}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}$ the image of the set of initial datum $\mathscr{R}^{>,t-t_s,T}_{\{q\},\omega,(s_i)_i} \cap \{T \text{ is the clusturing tree}\}$ by $(X_{2,n_{k+2}}, V_{n_{k+2}}) \to (\tau_{[n_{k+2}-1]}, \nu_{[n_{k+2}-1]}, V_{n_{k+2}})$.

$$\int \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}} \frac{e^{-\mathscr{H}_{n_{k+2}}}}{(2\pi)^{\frac{n_{k+2}d}{2}}}\, \mathrm{d}X_{2,n_{k+2}} dV_{n_{k+2}}$$
$$= \frac{1}{(\mu\mathfrak{d})^{n_{k+2}-1}} \sum_T \int_{\mathfrak{T}_{\underline{n}_{k+2}} \times \mathfrak{G}^{>,t-t_s,T}_{\{q\},\omega,(s_i)_i}} \prod_{i=1}^{n_{k+2}-1} |(\mathrm{v}^\varepsilon_{q_i}(\tau_i) - \mathrm{v}^\varepsilon_{q_i'}(\tau_i))\cdot\nu_i|\, \mathrm{d}\nu_i\, \mathrm{d}\tau_i M^{\otimes n_{k+2}} dV_{n_{k+2}}.$$

If the first recollision involves particles q and q' at time $\tau_{\mathrm{rec}} \in]\tau, c, \tau_{c+1}[$, we consider $\omega \subset [n_{k+2}]$ the connected components of $\{q, q'\}$ in the collision graph on time interval $[0, \tau_{\mathrm{rec}})$ (it only depends on c). As before the first recollision, the pseudotrajectory $\mathrm{Z}^\varepsilon_\omega(\tau)$ and its formal limit $\mathrm{Z}_\omega(\tau)$ are closed up to a translation (thanks to Lemma 6.3): there exists a $y_0 \in \mathbb{T}$ such that

$$\forall \tau \in [0, \tau_{\mathrm{rec}}],\ |\mathrm{X}^0_\omega(\tau) - \mathrm{tr}_{y_0}\mathrm{X}^\varepsilon_\omega(\tau)| \le \sum_{i=1}^{n_{k+2}-1} \frac{2n_{k+2}\mathbb{V}\varepsilon}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}.$$

Hence, if there is a recollision,

$$|\mathrm{x}^0_q(\tau_{\mathrm{rec}}) - \mathrm{x}^0_{q'}(\tau_{\mathrm{rec}})| \le \varepsilon + \sum_{i=1}^{n_{k+2}-1} \frac{2n_{k+2}\mathbb{V}\varepsilon}{\left|\mathrm{v}_{q_i}(\tau_i^-) - \mathrm{v}_{q_i}(\tau_i^-)\right|}. \tag{7.6}$$

We can only study the limiting flow and define a recollision as "there exists a time τ_{rec} such that (7.6) is verified: we have

$$\mathfrak{T}_{\underline{n}_{k+2}} \times \mathfrak{G}^{>,t-t_s,T}_{\{q\},\omega,(s_i)_i} \subset \mathfrak{T}_{\underline{n}_{k+2}} \times \mathfrak{G}^0_T \setminus \mathfrak{T}_{\underline{n}_{k+2}} \times \mathfrak{G}^\varepsilon_T.$$

Using the Lemma B.2, we get

$$\begin{aligned} \int \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}} \frac{e^{-\mathscr{H}_{n_{k+2}}}}{(2\pi)^{\frac{n_{k+2}d}{2}}}\, \mathrm{d}X_{2,n_{k+2}} dV_{n_{k+2}} &\le \frac{(Cn_{k+2})^{n_{k+2}}}{(\mu\mathfrak{d})^{n_{k+2}-1}} t^{n_k-1}\theta^{(n_{k+2}-n_k-1)_+}\varepsilon^{1/4} \\ &\le \frac{(Cn_{k+2})^{n_{k+2}}}{(\mu\mathfrak{d})^{n_{k+2}-1}} t^{n_k-1}\theta^{(n_{k+2}-n_k-1)_+}\delta^2\varepsilon^{1/12} \end{aligned} \tag{7.7}$$

using that $\delta = \varepsilon^{1/12}$.

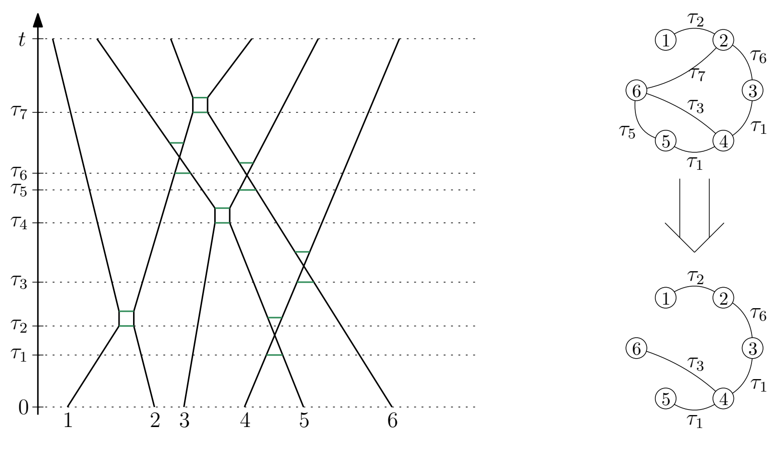

FIGURE 9. Example of construction of a clustering tree.

□

We obtain the expected result by summing on

$$(s_i)_{i\leq n_{k+2}-1} \in \{\pm 1\}^{n_{k+2}-1},\ \omega\subset [n_{k+2}],\ q\in\omega$$

and dividing by $n_{k+2}!$. □

Proof of (7.4). We use first the same bound as in the previous Section

$$(7.8)\quad \left|\Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h] \circledast_m \Psi^{>,t-t_s}_{\underline{n}_{k+2}}[h](Z_{2n_{k+2}-m})\right| \leq \frac{\|h\|^2}{(n_{k+2}!)^2}\frac{(n_{k+2}-m)!^2 m!}{(2n_{k+2}-m)!}\sum_{\substack{\bar\omega\cup\bar\omega'=[2n_{k+2}-m]\\ |\bar\omega|=|\bar\omega'|=n_{k+2}}}\ \sum_{\substack{q\in\omega\subset\bar\omega\\ |\omega|=n_{k+1}\\ (s_i)_{i\leq n_{i+2}-1}}}\ \sum_{\substack{q'\in\omega'\subset\bar\omega'\\ |\omega'|=n_{k+1}\\ (s'_i)_{i\leq n_{i+2}-1}}} \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}}(Z_{\bar\omega})) \times \mathbb{1}_{\mathfrak{n}(k'\delta)=n_k}\mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q'\},\omega',(s'_i)_i}}(Z_{\bar\omega'}).$$

where $\mathfrak{n}(\theta)$ is the number of particles at time θ in the pseudotrajectory $\mathrm{Z}(\tau)$. Note that the formula is invariant under translation. Without loss of generality, we can suppose that We can then fix $x_1=0$ and integrate with respect to the other variables.

Using the same strategy as in the proof of (5.3), we have

$$\int \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q'\},\omega',(s'_i)_i}}(Z_{\bar\omega'})e^{-\frac12\mathscr{H}_{2n_{k+2}-m}}\,\mathrm{d}Z_{\bar\omega'\setminus\bar\omega} \leq C\left(\frac{C}{\mu\mathfrak{d}}\right)^{n_k-m} t^{n_{k+2}-m}(2n_{k+2}-m)^{n_{k+2}-m}.$$

The sum on the remaining particles is estimated using (7.7)

$$\int \mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q'\},\omega',(s'_i)_i}}(Z_{\bar\omega'})\mathbb{1}_{\mathscr{R}^{>,t-t_s}_{\{q\},\omega,(s_i)_i}}(Z_\omega)\frac{e^{-\mathscr{H}_{2n_{k+2}-m}}}{(2\pi)^{\frac{(2n_{k+2}-m)d}{2}}}\,\mathrm{d}X_{1,2n_{k+2}-m}dV_{2n_{k+2}-m} \leq \frac{(C2n_{k+2})^{2n_{k+2}-m}}{(\mu\mathfrak{d})^{2n_{k+2}-m-1}}\delta^2\theta^{(n_{k+2}-n_k-3)_+}t^{n_k+n_{k+2}+9+d}$$

We obtain the expected result by combining the two estimations, summing on the possible parameters $((s_i)_i,\bar\omega,\omega,q)$ and $((s'_i)_i,\bar\omega',\omega',q')$ and then dividing by $(n_{k+2})!^2$. □

8. Estimation of the local recollisions

In the present section we discuss $G^{\mathrm{rec},2}_\varepsilon(t)$ defined by

$$G^{\mathrm{rec},2}_\varepsilon(t) = \sum_{\substack{0\leq k\leq K-1\\ 1\leq k'\leq K'}}\ \sum_{\substack{(n_j)_{j\leq k}\\ 0\leq n_j-n_{j-1}\leq 2^j}}\ \sum_{n_{k+2}\geq n_{k+1}\geq n_k}\mathbb{E}_\varepsilon\Bigg[\frac{1}{\sqrt{\mu}}\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon} \times\sum_{\underline{i}_{n_{k+2}}}\Phi^{>,\delta}_{n_{k+1},n_{k+2}}\Psi^{,t-t_s-\delta}_{\underline{n}_{k+1}}[h]\left(\mathbf{Z}_{\underline{i}_{n_{k+2}}}(t_s)\right)\Bigg]$$

for $t_s := t-k\theta-k'\delta$.

We will prove the following bound:

Proposition 8.1. *For $\varepsilon>0$ small enough, we have*

$$(8.1)\quad \left|G^{rec,2}_\varepsilon(t)\right| \leq C\|h\|_0\|g\|_0 K2^{K^2}(C\tfrac{t}{\mathfrak{d}})^{2^{K+1}}\varepsilon^{\frac{\mathfrak{a}}{2}}.$$

In the following, we denote

$$\bar\Phi^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}}) := \frac{1}{(n_{k+2})!}\sum_{\sigma\in\mathfrak{S}_{n_{k+2}}}\Phi^{>,\delta}_{n_{k+1},n_{k+2}}\Psi^{,t-t_s-\delta}_{\underline{n}_{k+1}}[h]\left(Z_{\sigma[n_{k+2}]}\right)$$

The aim of this part is to prove the following bound on $\Phi^{k'}_{\underline{n}_{k+2},p}$:

Proposition 8.2. *Fix $n_1\leq\cdots\leq n_{k+2}\leq p$. For $m\in\{1,\cdots p\}$ we have for $x_1=0$*

$$(8.2)\quad \int\sup_{y\in\mathbb{T}}\left|\bar\Phi^{k'}_{\underline{n}_{k+2}}(\mathrm{tr}_y\,Z_{n_{k+2}})\right|\frac{e^{-\mathscr{H}_{n_{k+2}}(Z_{n_{k+2}})}}{(2\pi)^{\frac{dn_{k+2}}{2}}}\,\mathrm{d}Z_{2,n_{k+2}}\,\mathrm{d}v_1 \leq \frac{\|h\|_0}{(\mu\mathfrak{d})^{n_{k+2}-1}}C^{n_{k+2}}\delta^2\varepsilon^{\mathfrak{a}}\theta^{(n_{k+2}-n_k-2)_+}t^{n_k-1},$$

$$(8.3)\quad \int \sup_{y\in\mathbb{T}} \left|\bar{\Phi}^{k'}_{\underline{n}_{k+2}} \circledast_m \bar{\Phi}^{k'}_{\underline{n}_{k+2}}(\mathrm{tr}_y Z_{2n_{k+2}-m}) \frac{e^{-\mathscr{H}_{2n_{k+2}-m}}}{(2\pi)^{\frac{d(2n_{k+2}-m)}{2}}} \,\mathrm{d}Z_{2,2n_{k+2}-m}\,\mathrm{d}v_1 \right. \\ \leq \frac{\mu^{m-1}}{n_{k+2}^m}\left(\frac{\|h\|_0}{(\mu\mathfrak{d})^{n_{k+2}-1}}C^{n_{k+2}}\right)^2 \delta^2\varepsilon^{\mathfrak{a}}\theta^{(n_{k+2}-n_k-2)_+}t^{n_k-1+n_{k+2}}.$$

Using the estimations (8.2) and (8.3), one obtains

$$\left|\mathbb{E}_\varepsilon\left[\mu^{-\frac12}\sum_{\underline{i}_{n_{k+2}}}\Phi^{k'}_{\underline{n}_{k+2}}(\mathbf{Z}_{\underline{i}_{n_{k+2}}}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right)\Bigg| \leq \|g\|_0\|h\|_0 C^{n_{k+2}}\Big((\tfrac{\delta}{\mathfrak{d}})^2(\tfrac{\theta}{\mathfrak{d}})^{(n_{k+2}-n_k-2)_+}(\tfrac{t}{\mathfrak{d}})^{n_k-1}\varepsilon^{\frac{2\mathfrak{a}+1}{2}} \\ +\Big((\tfrac{\delta}{\mathfrak{d}})^2(\tfrac{\theta}{\mathfrak{d}})^{(n_{k+2}-n_k-2)_+}(\tfrac{t}{\mathfrak{d}})^{n_{k+2}+n_k-1}\varepsilon^{\mathfrak{a}}\Big)^{\frac12}\Big) \\ \leq \delta\varepsilon^{\frac{\mathfrak{a}}{2}}\|h\|_0\|g\|_0 C^{n_{k+2}}(\tfrac{t^2\theta}{\mathfrak{d}^3})^{\frac{(n_{k+2}-n_k-2)_+}{2}}(\tfrac{t}{\mathfrak{d}})^{n_k},$$

as $\varepsilon^{\frac12}/\mathfrak{d}\to 0$.

Using that $\frac{Ct\theta}{\mathfrak{d}^2}\leq 1$ and $K'\delta=\theta\leq 1$, we can sum on k, k' and $\underline{n}_{k+2}$

$$\left|G^{\mathrm{rec},2}_\varepsilon(t)\right| \leq \sum_{\substack{1\leq k\leq K-1\\ 1\leq k'\leq K'}}\ \sum_{\substack{n_1\leq\cdots\leq n_k\\ n_j-n_{j-1}\leq 2^j}}\ \sum_{n_{k+2}\geq n_{k+1}\geq n_k}\|g\|_0\|h\|_0\delta\varepsilon^{\frac{\mathfrak{a}}{2}}(\tfrac{Ct}{\mathfrak{d}})^{n_k+d+}(\tfrac{Ct\theta}{\mathfrak{d}^2})^{\frac{(n_{k+2}-n_k-2)_+}{2}} \\ \leq \|g\|_0\|h\|_0 K'\delta\varepsilon^{\frac{\mathfrak{a}}{2}}K^{K^2}(\tfrac{Ct}{\mathfrak{d}})^{2^K+d+9} \\ \leq \|g\|_0\|h\|_0\varepsilon^{\frac{\mathfrak{a}}{2}}(\tfrac{C't}{2\mathfrak{d}})^{2^K+d+9}$$

This conclude the proof of (8.1).

Proof of (8.2). We recall that

$$\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}}) := \frac{1}{(n_{k+2})!}\sum_{\sigma\in\mathfrak{S}_{n_{k+2}}}\Phi^{>,\delta}_{n_{k+1},n_{k+2}}\Psi^{,t-t_s-\delta}_{\underline{n}_{k+1}}[h]\left(Z_{\sigma[n_{k+2}]}\right)$$

In $\Psi^{0,t-t_s-\delta}_{\underline{n}_{k+1}}\Phi^{0,\delta}_{n_{k+1},n_{k+2}}[h]\left(Z_{[n_{k+2}]}\right)$ we see three sets of indices:

- 1 the last particle,
- $[1,n_{k+1}]$ the set of particles in "final" tree pseudotrajectory development,
- $[n_{k+1}+1,n_{k+2}]$ the particles added in the first time interval.

Any permutation σ which sends $[1,n_{k+1}]$ and $[n_{k+1}+1,n_{k+2}]$ onto themselves stabilizes the function $\Psi^{0,t-t_s-\delta}_{\underline{n}_{k+1}}\Phi^{0,\delta}_{n_{k+1},n_{k+2}}[h]$. Hence, $\bar{\Phi}^{k'}_{\underline{n}_{k+2},p}(Z_p)$ is equal to

$$\frac{(n_k-1)!(n_{k+2}-n_{k+1})!}{(n_k+2)!}\sum_{\substack{\omega_1\sqcup\omega_2=[n_{k+2}]\\ |\omega_1|=n_{k+1}\\ q_1\in\omega_1}}\Phi^{>,\delta}_{n_{k+1},n_{k+2}}\Psi^{,t-t_s-\delta}_{\underline{n}_{k+1}}[h]\left(z_{q_1},Z_{\omega_1\setminus\{q_1\}},Z_{\omega_2}\right).$$

Let us develop $\Psi^{0,t-t_s-\delta}_{\underline{n}_{k+1}}\Phi^{0,\delta}_{n_{k+1},n_{k+2}}[h]$. For $(s_i)_i\in\{\pm1\}^{n_{k+1}-1}$, (ω_1,ω_2) a partition of $[n_{k+2}]$ and $(\lambda_1,\cdots,\lambda_{\mathfrak{l}})$ a partition of $[n_{k+2}]$ with $\omega_1\subset\lambda_1$, we define the pseudotrajectory $\bar{\mathsf{Z}}(\tau,Z_{n_{k+2}},q_1,\omega_1,\omega_2,(s_i)_i,(\lambda_j)_j)$ by

- for $\tau\leq\delta$,

$$\bar{\mathsf{Z}}(\tau):=\mathsf{Z}(\tau,Z_{\omega_1},(\lambda_j)_j)$$

- for $\tau>\delta$, the particle of ω_3 are removed and

$$\bar{\mathsf{Z}}_{\omega_1}(\tau):=\mathsf{Z}(\tau-\delta,\bar{\mathsf{Z}}_{\omega_1}(\delta),\{q_1\},(s_i)_i).$$

Then $\bar\Phi^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}})$ is equal to

$$\frac{1}{(n_{k+2})!}\sum_{\substack{\omega_1\sqcup\omega_2=[n_{k+2}]\\|\omega_1|=n_{k+1}\\q_1\in\omega_1}}\sum_{(s_i)_i}\left(\sum_{\mathfrak{l}=1}^{n}\sum_{\lambda_1\supset\omega_1}\sum_{\substack{(\lambda_2,\cdots,\lambda_{\mathfrak{l}})\\\in\mathscr{P}^{\mathfrak{l}-1}_{\omega_2\setminus\lambda_1}}}\right)h\left(\bar{\mathsf{z}}_{\omega_1}(k\theta+k'\delta)\right)$$
$$\times\,\mathbb{1}_{\substack{\bar{\mathsf{Z}}(\cdot)\text{ has a}\\\text{pathology}\\\text{on }[0,\delta]}}\left(\prod_{i=1}^{n_{k+2}-1}s_i\mathbb{1}_{\mathscr{R}^{0,k\theta+(k-1)\delta}_{\omega_1,(s_i)_i}}(\bar{\mathsf{Z}}_{\omega_2}(\delta))\prod_{i=1}^{k}\mathbb{1}_{\mathfrak{n}(t-i\theta)=n_i}\right)$$
$$\times\left(\varphi_l(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathfrak{l}}})\Delta^{[m]}_{|\lambda_1|}(Z_{\lambda_1})\prod_{i=2}^{\mathfrak{l}}\Delta_{|\lambda_i|}(Z_{\lambda_i})\right).$$

The functions φ and Δ are defined in Definitions 3.0.1 and 3.0.2 (they are define on a time t, here it is replaced by a time δ).

The function $\varphi_l(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathfrak{l}}})$ can be bounded by the Penrose's tree inequality (see for example [BGSRS20, Jan]),

$$|\varphi_l(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathfrak{l}}})|=\left|\sum_{C\in\mathscr{C}([\mathfrak{l}])}\prod_{(i,j)\in E(C)}-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right|\leq\sum_{T\in\mathscr{T}([\mathfrak{l}])}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}.$$

The set $\mathscr{T}([\mathfrak{l}])$ is the set of minimally connected graph with vertices $[\mathfrak{l}]$.

We explain now how to take advantage of the pathology of $\bar{\mathsf{Z}}(\cdot)$.

Definition 8.2.1. *For $r\geq 3$, we define the set $\mathscr{O}_r$ as*

$$\mathscr{O}_r:=\Big\{Z_r\in\mathbb{D}^r,\ \exists(\varpi_1,\cdots,\varpi_{\mathfrak{l}}),\ \text{the collision graph of }\mathsf{Z}_r(\cdot,Z_z,(\varpi_1,\cdots,\varpi_{\mathfrak{l}}))\ \textit{on }[0,\delta]\text{ is connected and the pseudotrajectory has a pathology}\Big\}.\tag{8.4}$$

We recall that a pathology can be an overlap, a multiple interaction or a recollision (see the Definition 3.1.1).

For $r=2$, we define

$$\mathscr{O}_2:=\{|x_1-x_2|\leq\varepsilon\}\cup\{|(x_1-x_2)+\delta(v_1-v_2)|\leq\varepsilon\}.\tag{8.5}$$

Finally for $\varpi\subset[n_{k+2}]$, the set $\mathscr{O}_\varpi$ is defined as

$$\mathscr{O}_\varpi:=\left\{Z_{n_{k+2}}\in\mathbb{D}^{n_{k+2}},\ Z_\varpi\in\mathscr{O}_{|\varpi|}\right\}.\tag{8.6}$$

The $\mathscr{O}_\varpi$ allows to control the recollision condition

$$\mathbb{1}_{\substack{\bar{\mathsf{Z}}(\cdot)\text{ has a}\\\text{pathology}\\\text{on }[0,\delta]}}\leq\sum_{\varpi\subset[n_{k+2}]}\mathbb{1}_{\mathscr{O}_\varpi}.$$

This leads to the following bound on $\left|\bar\Phi^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}})\right|$:

$$\frac{\|h\|_0}{(n_{k+2})!}\sum_{\substack{\omega_1\sqcup\omega_2=[n_{k+2}]\\|\omega_1|=n_{k+1}\\q_1\in\omega_1}}\sum_{\varpi\subset[n_{k+2}]}\sum_{(s_i)_i}\left(\sum_{\mathfrak{l}=1}^{n}\sum_{\lambda_1\supset\omega_1}\sum_{\substack{(\lambda_2,\cdots,\lambda_{\mathfrak{l}})\\\in\mathscr{P}^{\mathfrak{l}-1}_{\omega_2\setminus\lambda_1}}}\mathbb{1}_{\mathscr{R}^{0,k\theta+(k-1)\delta}_{\{q_1\},(s_i)_i}}(\bar{\mathsf{Z}}_{\omega_1}(\delta))\mathbb{1}_{\mathfrak{n}(k'\delta)=n_k}\right.$$
$$\left.\times\,\mathbb{1}_{\mathscr{O}_\varpi}\sum_{T\in\mathscr{T}([\mathfrak{l}])}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\Delta^{[n_{k+1}]}_{|\lambda_1|}(Z_{\lambda_1})\prod_{i=2}^{\mathfrak{l}}\Delta_{|\lambda_i|}(Z_{\lambda_i})\right).\tag{8.7}$$

Note that the right hand-side is invariant under translation. Thus one can fix $x_1=0$ and integrate with respect to the other variables.

We introduce a partition to control the pseudo-trajectory in the time interval $[0,\delta]$.

Definition 8.2.2 (Possible clusters)**.** *Given $Z_{n_{k+2}}\in\mathbb{D}^{n_{k+2}}$, we construct the graph G with vertices $[n_{k+2}]$. The pair (i,j) is an edges of G if and only if there exists $\tilde\omega\subset[n_{k+2}]$ and $(\tilde\lambda_1,\cdots,\tilde\lambda_\ell)$ a partition of $\tilde\omega$ such that the collision graph of $\mathsf{Z}(\cdot,Z_{\tilde\omega},\tilde\lambda_1,\cdots,\tilde\lambda_\ell)$ on time interval $[0,\delta]$ is connected. We introduce $\underline\rho:=(\rho_1,\cdots,\rho_{\mathbf{r}})$ the* possible cluster partition *as the set of the connected components of G.*

We define $\mathscr{D}_\varepsilon^{\underline{\rho}} \subset \mathbb{D}^{n_{k+2}}$ as the set such that $\underline{\rho}$ is the possible cluster partition. The $\big(\mathscr{D}_\varepsilon^{\underline{\rho}}\big)_{\underline{\rho}}$ form a partition of $\mathbb{D}^{n_{k+2}}$.

By definition of the potential cluster, a particle cannot interact with a particle of an other cluster for any time in $[0,\delta]$. Thus the systems ρ^i are isolated on $[0,\delta]$ and all the dynamics in $[0,\delta]$ is encode inside the (ρ_i).

The parametrization of the pseudotrajectories is changed to a more adapted one. There exists a ρ_i containing ϖ. With a little lost of symmetry one can suppose that it is ρ_1. In the same way for any λ_j with $j \neq 1$ there exists some ρ_i containing λ_j. For any ρ_i

- $\underline{\omega}^i := (\omega_1^i, \omega_2^i)$ the partition of ρ_i defined by $\omega_j^i := \omega_j \cap \rho_i$, note that the set ω_1^i cannot be empty,
- $\underline{\lambda}^i := \{\lambda_1^i := \lambda_1 \cap \rho_i\} \cup \{\lambda_j \text{ for } j \geq 2 \text{ with } \lambda_j \subset \rho_i\}$ a partition of ρ_i,
- for $i \geq 1$, $\mathfrak{p}_i := (\underline{\omega}^i, \underline{\lambda}^i)$,
- $\mathfrak{p}_1 := (\underline{\omega}^1, \underline{\lambda}^1, \varpi)$.

The set of possible $\mathfrak{p}_i$ is denoted $\mathfrak{P}(\rho_i)$. Because ρ_i is of size at most γ, there exists a constant C_γ depending only on γ such that $|\mathfrak{P}(\rho_i)| \leq C_\gamma$. For a fix partition $\underline{\rho}$, the map $(\underline{\omega}, \varpi, \underline{\lambda}) \mapsto (\mathfrak{p}_i)_i$ is onto.

The possible cluster partition also contains the overlap: if we denote two dynamical clusters λ_j and $\lambda_{j'}$ with $j, j' \geq 2$, there exists a ρ_i containing both, and if $\lambda_j \subset \rho_i$ has an overlap with λ_1, then λ_j has an overlap with λ_1^i. This last property allows us to rewrite the overlap cumulant: for any $Z_{n_{k+2}}$ in $\mathscr{D}_\varepsilon^{\underline{\rho}}$,

$$\Big|\psi_{\mathbf{l}}\big(Z_{\lambda_1}, \cdots, Z_{\lambda_{\mathbf{l}}}\big)\Big| \leq \sum_{T \in \mathscr{T}([\mathbf{l}])} \prod_{(i,j) \in E(T)} \mathbb{1}_{\lambda_i \overset{\circ}{\sim} \lambda_j}$$

$$\leq \prod_{i=1}^{\mathbf{r}} \sum_{T_i \in \mathscr{T}([|\underline{\lambda}^i|])} \prod_{(j,j') \in E(T_i)} \mathbb{1}_{\lambda_j^i \overset{\circ}{\sim} \lambda_{j'}^i} \leq \prod_{i=1}^{\mathbf{l}} \Big|\mathscr{T}([|\underline{\lambda}^i|])\Big|.$$

The right hand-side is bounded using that

$$\Big|\mathscr{T}([|\underline{\lambda}^i|])\Big| \leq |\underline{\lambda}^i|^{|\underline{\lambda}^i|-2} \leq |\rho_i|^{|\rho_i|}$$

(see section 2 of [BGSRS20]). As the symmetric conditioning imposes that $|\rho_i| \leq \gamma$, the cumulants $\Big|\psi_{\mathbf{l}}\big(Z_{\lambda_1}, \cdots, Z_{\lambda_{\mathbf{l}}}\big)\Big|$ are smaller than $\gamma^{n_{k+2}}$.

We have now the following bound

$$\big|\bar{\Phi}^{k'}_{\underline{n}_{k+2},p}(Z_p)\big| \leq \frac{\gamma^{n_{k+2}} \|h\|_0}{(n_{k+2})!} \sum_{q_1=1}^{n_{k+2}} \sum_{\mathbf{r}=1}^{n_{k+2}} \sum_{\underline{\rho} \in \mathscr{P}^{\mathbf{r}}_{n_{k+2}}} \sum_{\substack{(s_i)_i \\ \underline{\mathfrak{p}} \in \prod_i \mathfrak{P}(\rho_i)}} \mathbb{1}_{\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i)_i}}(Z_{n_{k+2}}) \prod_{i=1}^{\mathbf{r}} \Delta_{\mathfrak{p}_i}(Z_{\rho_i}) \tag{8.8}$$

where we denote

$$\Delta_{\mathfrak{p}_1}(Z_{\rho_1}) := \mathbb{1}_{\mathscr{O}_\varpi} \mathbb{1}_{Z_{\rho_1} \text{ form a possible cluster}},$$

$$\forall i \geq 2,\ \Delta_{\mathfrak{p}_i}(Z_{\rho_i}) := \mathbb{1}_{Z_{\rho_i} \text{ form a possible cluster}} \quad \text{and}$$

$$\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{(s_i)_i} := \Big\{ Z_p \in \mathscr{D}_\varepsilon^{\underline{\rho}},\ \bar{\mathsf{Z}}_{\omega_1}(\delta) \in \mathscr{R}^{0,k\theta+(k-1)\delta}_{\{q_1\},(s_i)_i} \Big\}.$$

The same method than in [BGSRS22c] is used to control the condition $\mathbb{1}_{\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i)_i}}$.

For a pseudotrajectory $\bar{\mathsf{Z}}_{n_{k+2}}(\tau)$, consider its collision graph $\mathscr{G}^{[0,t-t_s]}_{\omega_1 \cup \omega_2}$. Then, we can construct the graph G by identifying in $\mathscr{G}^{[0,t-t_s]}_{\omega_1 \cup \omega_2}$ the particles in a same cluster ρ_i. Finally we can construct the *clustering trees* $T^> := (\nu_i, \bar{\nu}_i)_{1 \leq i \leq \mathbf{r}-1}$ where the i-th clustering collision in G happens between cluster ρ_{ν_i} and $\rho_{\bar{\nu}_i}$.

We need to count the number of clustering collisions of $T^>$ happening between time δ and time $k'\delta$. If $\mathbf{r} > n_k$, all the $\mathbf{r}-1$ collisions in $T^>$ cannot correspond to the $n_k - 1$ collisions of the time interval $[k'\delta, \theta]$. Thus, at least $(\mathbf{r} - n_k)_+$ collisions happen during $[\delta, k'\delta] \subset [0, 2\theta]$.

One needs a different representation of collision graphs. Let L_0 be equal to $\{\{1\}, \cdots, \{\mathbf{r}\}\}$. The L_i and $(\nu_{(i)}, \bar{\nu}_{(i)})$ are constructed sequentially: suppose that $L_{i-1} = (c_1, \cdots, c_l)$, the (c_j) forming a partition of $[1, r]$. The i-th collision happens between cluster $\nu_i \in c_a$ and $\bar{\nu}_i \in c_b$. Then:

- $L_i := \big(L_{i-1} \setminus \{c_a, c_b\}\big) \cup \{c_a \cup c_b\}$,
- $\{\nu_{(i)}, \bar{\nu}_{(i)}\} := \{c_a, c_b\}$ with $\max \nu_{(i)} < \max \bar{\nu}_{(i)}$.

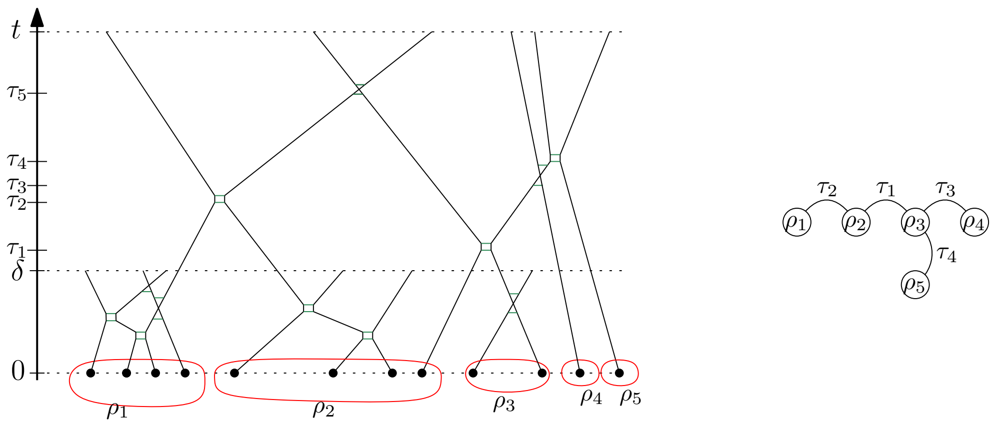

FIGURE 10. Example of construction of the clustering stets.

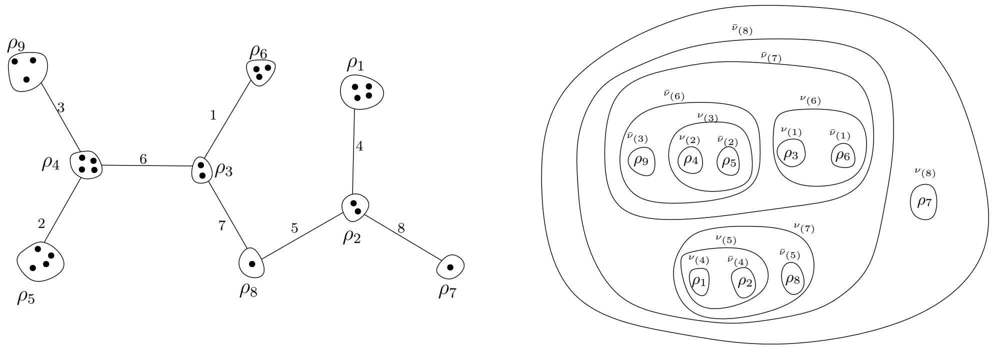

FIGURE 11. An example of construction of the representation $(\nu_{(i)}, \bar{\nu}_{(i)})_i$ from a clustering graph.

The $(\nu_{(i)}, \bar{\nu}_{(i)})_i$ define a partition of $\mathscr{T}^{>}([\mathbf{r}])$ (the set of ordered trees on $[1, \mathbf{r}]$).

We performed the following change of variables:

$$\forall i \in \{1, \cdots, \mathbf{r}-1\},\ \hat{x}_i := x_{\min \nu_{(i)}} - x_{\min \bar{\nu}_{(i)}},\ \tilde{X}_i := \mathrm{tr}_{-x_{\min \rho_i}} X_{\rho_i},$$

$$X_{2,n_{k+2}} \mapsto (\hat{x}_1 \cdots, \hat{x}_{\mathbf{r}-1}, \tilde{X}_1, \cdots, \tilde{X}_{\mathbf{r}}).$$

The condition $\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}$ is integrated first with respect to $(\hat{x}_1, \cdots, \hat{x}_{\mathbf{r}-1})$ where relative positions inside a cluster $\tilde{X}_i$ are kept constant. The $(\Delta_{\mathfrak{p}_i})_i$ will be sum later with respect to the $(\tilde{X}_i)_i$.

Fix τ_{i+1} the time of the $(i+1)$-th clustering collision and the relative positions $\hat{x}_{i-1}, \cdots, \hat{x}_1$. We define the i-th clustering set

$$B_i := \bigcup_{\substack{q \in \omega_{2,\nu_{(i)}} \\ \bar{q} \in \omega_{1,\bar{\nu}_{(i)}}}} B_i^{q,\bar{q}}$$

$$\text{with} \qquad \omega_{1,\nu_{(i)}} := \bigcup_{j \in \nu_{(i)}} \omega_1^j, \qquad \omega_{1,\bar{\nu}_{(i)}} := \bigcup_{\bar{j} \in \bar{\nu}_{(i)}} \omega_2^{\bar{j}},$$

$$B_i^{q,\bar{q}} := \left\{ \hat{x}_i \;\middle|\; \exists \tau_i \in [0, \tau_{i+1} \wedge T_i],\ |\bar{\mathsf{x}}_{\bar{q}}(\tau_i) - \bar{\mathsf{x}}_{\bar{q}}(\tau_i)| = \varepsilon \right\}$$

and $T_i := 2\theta$ for the the $(\mathbf{r} - n_k)_+$ first collisions, t else. We used that $\omega_1 \cup \omega_2$ is the set of particles existing after time δ.

Up to time τ_i the curves $\bar{\mathsf{x}}_q$ and $\bar{\mathsf{x}}_{\bar{q}}$ are independent. Hence, we can perform the change of variables $\hat{x}_i \mapsto (\tau_i, \eta_i)$ where τ_i is the minimal collision time and

$$\eta_i := \frac{\bar{\mathsf{x}}_{\bar{q}}(\tau_i) - \bar{\mathsf{x}}_q(\tau_i)}{|\bar{\mathsf{x}}_{\bar{q}}(\tau_i) - \bar{\mathsf{x}}_q(\tau_i)|}.$$

The Jacobian of this diffeomorphism is $\varepsilon^{d-1} |(\bar{\mathsf{v}}_{\bar{q}}(\tau_i) - \bar{\mathsf{v}}_q(\tau_i)) \cdot \eta_i| \, d\tau_i \, d\eta_i$.

As the particles in $\omega_{1,\nu_{(i)}}$ and $\omega_{1,\bar\nu_{(i)}}$ are isolated during $[\delta,\tau_i]$, their energies are conserved . The sum of relative velocities can be bounded by

$$\begin{aligned}\sum_{\substack{q\in\omega_{1,\nu_{(i)}}\\ \bar q\in\omega_{1,\bar\nu_{(i)}}}} |\bar{\mathsf v}_{\bar q}(\tau_i)-\bar{\mathsf v}_q(\tau_i)| &\leq |\bar{\mathsf V}_{\omega_{1,\nu_{(i)}}}(\tau_i)|\,|\omega_{1,\nu_{(i)}}|^{1/2}|\omega_{1,\bar\nu_{(i)}}| + |\bar{\mathsf V}_{\omega_{1,\bar\nu_{(i)}}}(\tau_i)|\,|\omega_{1,\bar\nu_{(i)}}|^{1/2}|\omega_{1,\nu_{(i)}}|\\ &\leq \Big(|\omega_{1,\nu_{(i)}}|+|\bar{\mathsf V}_{\omega_{1,\nu_{(i)}}}(\tau_i)|^2\Big)\Big(|\omega_{1,\bar\nu_{(i)}}|+|\bar{\mathsf V}_{\omega_{1,\bar\nu_{(i)}}}(\tau_i)|^2\Big).\end{aligned}$$

Using the same method than in the proof of 5.5,

$$\frac12\left|\bar{\mathsf V}_{\omega_{1,\nu_{(i)}}}(\tau_i)\right|^2 \leq \mathscr H_{|\omega_{1,\nu_{(i)}}|}\left(\bar{\mathsf Z}_{\omega_{1,\nu_{(i)}}}(\delta)\right) \leq \mathscr H_{|\lambda_{1,\bar\nu_{(i)}}|}\left(\bar{\mathsf Z}_{\lambda_{1,\bar\nu_{(i)}}}(\delta)\right)$$

where we denote

$$\lambda_{1,\bar\nu_{(i)}} := \bigcup_{\bar\jmath\in\bar\nu_{(i)}} \lambda_1^j.$$

At time δ, the particles in two different clusters cannot interact (by definition of a possible cluster),

$$\mathscr H_{|\lambda_{1,\nu_{(i)}}|}(\bar{\mathsf Z}_{\lambda_{1,\nu_{(i)}}}(\delta)) = \sum_{j\in\nu_{(i)}} \mathscr H_{|\lambda_1^j|}(\bar{\mathsf Z}_{\lambda_1^j}(\delta)) = \sum_{j\in\nu_{(i)}} \mathscr H_{|\lambda_1^j|}(\bar{\mathsf Z}_{\lambda_1^j}(0)) \leq \sum_{j\in\nu_{(i)}} \mathscr H_{|\rho_j|}(Z_{\rho_j}).$$

We conclude that

$$\sum_{\substack{q\in\omega_{1,\nu_{(i)}}\\ \bar q\in\omega_{1,\bar\nu_{(i)}}}} |\bar{\mathsf v}_{\bar q}(\tau_i)-\bar{\mathsf v}_q(\tau_i)| \leq 4\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar\nu_i\in\bar\nu_{(i)}}} \Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big). \tag{8.9}$$

This gives the following bound on $|B_i|$ (using the Boltzmann-Grad scaling $\mu\mathfrak d\varepsilon^{d-1}=1$)

$$\begin{aligned}|B_i| &\leq \frac{C}{\mu\mathfrak d}\int_0^{t_{i+1}\wedge T_i} \mathrm d\tau_i \sum_{q,\bar q} |\bar{\mathsf v}_q(\tau_i)-\bar{\mathsf v}_{\bar q}(\tau_i)|\\ &\leq \frac{C}{\mu\mathfrak d}\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar\nu_i\in\bar\nu_{(i)}}} \Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big)\int_0^{t_{i+1}\wedge T_i} \mathrm d\tau_i.\end{aligned}$$

Permuting the product and the sum,

$$\begin{aligned}&\sum_{(\nu_{(i)},\bar\nu_{(i)})_i}\prod_{i=1}^{\mathbf r-1}\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar\nu_i\in\bar\nu_{(i)}}} \Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big)\\ &\qquad = \sum_{(\nu_i,\bar\nu_i)_i}\prod_{i=1}^{\mathbf r-1}\Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big).\end{aligned}$$

Using that

$$\forall a,b\in\mathbb N,\ \frac{(a+b)!}{a!b!}\leq 2^{a+b},$$

we have

$$\int_0^t \mathrm dt_{\mathbf r-1}\cdots\int_0^{t_2\wedge T_2}\mathrm dt_1 \leq \frac{t^{n_k\wedge\mathbf r-1}}{(n_k\wedge\mathbf r-1)!}\frac{\theta^{(\mathbf r-n_k)_+}}{((\mathbf r-n_k)_+)!} \leq 2^{n_k+2}\frac{t^{n_k\wedge\mathbf r-1}\theta^{(\mathbf r-n_k)_+}}{(\mathbf r-1)!}.$$

We can sum now on the clustering collisions:

$$\begin{aligned}\int \mathbb 1_{\mathscr R^{\rho,\mathfrak p}_{(s_i)_i}}\mathrm d\hat x_1\cdots\mathrm d\hat x_{\mathbf r-1} &\leq \sum_{(\nu_{(i)},\bar\nu_{(i)})}\int \mathrm d\hat x_1'\mathbb 1_{B_1}\int \mathrm d\hat x_2'\mathbb 1_{B_2}\cdots\int \mathrm d\hat x_{\mathbf r-1}\mathbb 1_{B_{\mathbf r-1}}\\ &\leq \left(\frac{C}{\mu\mathfrak d}\right)^{\mathbf r-1}\int_0^t \mathrm dt_{\mathbf r-1}\cdots\int_0^{t_2\wedge T_2}\mathrm dt_1\sum_{(\nu_i,\bar\nu_i)_i}\prod_{i=1}^{\mathbf r-1}\Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big)\\ &\leq \left(\frac{2C}{\mu\mathfrak d}\right)^{\mathbf r-1}\frac{t^{n_k\wedge\mathbf r-1}\theta^{(\mathbf r-n_k)_+}}{(\mathbf r-1)!}\sum_{(\nu_i,\bar\nu_i)_i}\prod_{i=1}^{\mathbf r-1}\Big(|\rho_{\nu_i}|+\mathscr H_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar\nu_i}|+\mathscr H_{|\rho_{\bar\nu_i}|}(Z_{\rho_{\bar\nu_i}})\Big).\end{aligned}$$

We can forget the order of the edges of $T^{>} = (\nu_i, \bar\nu_i)_i$, which gives a factor $\mathbf{r}!$. Secondly, denoting $d_i(G)$ the degree of the vertex i in a graph G and $\mathscr{T}([\mathbf{r}])$ the set of minimally (not oriented) connected graphs on $[1,\mathbf{r}]$, we can write the preceding inequality as

$$\int \mathbb{1}_{\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}} \,\mathrm{d}\hat x_1\cdots \mathrm{d}\hat x_{\mathbf{r}-1} \leq \left(\frac{2C}{\mu\mathfrak{d}}\right)^{\mathbf{r}-1} t^{n_k\wedge \mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+} \sum_{T\in\mathscr{T}([\mathbf{r}])}\prod_{i=1}^{r}\Big(|\rho_i| + \mathscr{H}_{|\rho_i|}(Z_{\rho_i})\Big)^{d_i(T)}.$$

For $A,B>0$, $x\in\mathbb{R}^+$, there exists a constant $C>0$ such that

$$(A+x)^B\, e^{-\frac{x}{4}} \leq \left(\tfrac{4B}{e}\right)^B e^{\frac{A}{4}}.$$

We use this inequality to bound

$$\begin{aligned}\int \mathbb{1}_{\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}}\, & e^{-\frac12 \mathscr{H}_{n_k+2}(Z_{n_k+2})}\,\mathrm{d}\hat x_1\cdots \mathrm{d}\hat x_{\mathbf{r}-1}\\ &\leq \left(\frac{C}{\mu\mathfrak{d}}\right)^{\mathbf{r}-1} t^{n_k\wedge r-1}\theta^{(\mathbf{r}-n_k)_+}\sum_{T\in\mathscr{T}([\mathbf{r}])}\prod_{i=1}^{\mathbf{r}}\Big(|\rho_i|+\mathscr{H}_{|\rho_i|}(Z_{\rho_i})\Big)^{d_i(T)} e^{-\frac12\sum_{i=1}^{\mathbf{r}}\mathscr{H}_{|\rho_i|}(Z_{\rho_i})}\\ &\leq \tilde C^{n_k+2}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mu\mathfrak{d})^{\mathbf{r}-1}}\sum_{T\in\mathscr{T}([\mathbf{r}])}\prod_{i=1}^{\mathbf{r}} d_i(T)^{d_i(T)}\,.\end{aligned}$$

As the sum of the $d_i(T)$ is equal to $2\mathbf{r}-2$, we have by convexity of $x\mapsto x\log x$

$$\sum_{i=1}^{\mathbf{r}} d_i(T)\log d_i(T) \leq \mathbf{r}\frac{\sum_{i=1}^{\mathbf{r}} d_i(T)}{\mathbf{r}}\log\frac{\sum_{i=1}^{\mathbf{r}} d_i(T)}{\mathbf{r}} \leq (2\mathbf{r}-2)\log 2$$

and $|\mathscr{T}([\mathbf{r}])|$ is equal to $\mathbf{r}^{\mathbf{r}-2}$,

$$\begin{aligned}\int \mathbb{1}_{\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}}\, & e^{-\frac12 \mathscr{H}_{n_k+2}(Z_{n_k+2})}\,\mathrm{d}\hat x_1\cdots \mathrm{d}\hat x_{\mathbf{r}-1}\\ &\leq \left(\frac{C}{\mu\mathfrak{d}}\right)^{\mathbf{r}-1} t^{n_k\wedge r-1}\theta^{(\mathbf{r}-n_k)_+}\sum_{T\in\mathscr{T}([\mathbf{r}])}\prod_{i=1}^{\mathbf{r}}\Big(|\rho_i|+\mathscr{H}_{|\rho_i|}(Z_{\rho_i})\Big)^{d_i(T)} e^{-\frac12\sum_{i=1}^{\mathbf{r}}\mathscr{H}_{|\rho_i|}(Z_{\rho_i})}\\ &\leq \tilde C'^{n_k+2}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mu\mathfrak{d})^{\mathbf{r}-1}}(\mathbf{r}-1)!\,.\end{aligned}$$

We can integrate now the condition $\Delta_{\mathfrak{p}_i}(Z_{\rho_i})$. The particles in Z_{ρ_i} have to form a possible cluster. Because clusters are of size at most γ,

$$\int_{\mathbb{T}^{|\rho_1|-1}\times(\mathbb{R}^d)^{|\rho_1|}}\Delta_{\mathfrak{p}_1}(Z_{\rho_1})\frac{e^{-\frac12\mathscr{H}_{|\rho_1|}(Z_{\rho_1})}}{(2\pi)^{d|\rho_1|/2}}\,\mathrm{d}\tilde X_i\,\mathrm{d}V_{\rho_1} \leq \frac{C_r\delta^{\min\{2,|\rho_1|-1\}}}{(\mu\mathfrak{d})^{|\rho_1|-1}}\varepsilon^{\mathfrak{a}}, \tag{8.10}$$

$$\int_{\mathbb{T}^{|\rho_i|-1}\times(\mathbb{R}^d)^{|\rho_i|}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})\frac{e^{-\frac12\mathscr{H}_{|\rho_i|}(Z_{\rho_i})}}{(2\pi)^{d|\rho_i|/2}}\,\mathrm{d}\tilde X_i\,\mathrm{d}V_{\rho_i} \leq C_\gamma\left(\frac{\delta}{\mathfrak{d}\mu}\right)^{|\rho_i|-1}, \tag{8.11}$$

The second inequality is a clustering estimation, similar to the ones threaten in the proof of (5.2). In the first inequality we use recollision estimates as in the proof of (7.3). The proofs are given in Appendix B.4.

Integrating the Δ_i leads to

$$\begin{aligned}\int \mathbb{1}_{\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}}(Z_{n_k+2})&\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})M^{\otimes n_k+2}(V_{n_k+2})\,\mathrm{d}X_{2,n_k+2}\,\mathrm{d}V_{n_k+2}\\ &\leq (\mathbf{r}-1)!C^{n_k+2}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mu\mathfrak{d})^{\mathbf{r}-1}}\prod_{i=1}^{\mathbf{r}}\left(\frac{\delta}{\mathfrak{d}\mu}\right)^{|\rho_i|-1}\frac{\delta^{\max 2,|\rho_1|-1}}{(\mu\mathfrak{d})^{|\rho_1|-1}}\varepsilon^{\mathfrak{a}}.\end{aligned}$$

Any particle removed at time δ has a clustering collision during $[0,\delta]$. Therefore $\sum_{i=1}^{\mathbf{r}}(|\rho_i|-1)$ is bigger than $n_{k+2}-n_k$. In addition we have chosen θ bigger than δ so

$$\int \mathbb{1}_{\mathscr{R}^{\rho,\mathfrak{p}}_{(s_i)_i}}(Z_{n_{k+2}})\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})\frac{e^{-\mathscr{H}_{n_{k+2}}(Z_{n_{k+2}})}}{(2\pi)^{\frac{dn_{k+2}}{2}}}\,\mathrm{d}X_{2,n_{k+2}}\,\mathrm{d}V_{n_{k+2}} \leq (\mathbf{r}-1)!\frac{C^{n_{k+2}}}{(\mu\mathfrak{d})^{n_{k+2}-1}}t^{n_k-1}\theta^{(n_{k+2}-n_k-2)_+}\delta^2\varepsilon^{\mathfrak{a}}.$$

We sum now on the parameters $(s_i)_i$ and $(\mathfrak{p}_i)$. Because size of possible clusters are bounded by γ, the $|\mathfrak{P}(\rho_i)|$ are smaller than some $C_\gamma>0$ depending only on γ. The number of collision parameters $(s_i)_i$ is equal to $2^{n_{k+2}}$ and

$$\int \left|\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}})\right|M^{\otimes n_{k+2}}\,\mathrm{d}X_{2,n_{k+2}}\,\mathrm{d}V_{n_{k+2}} \leq \frac{\|h\|(CC_\gamma 4\gamma)^{n_{k+2}}}{n_{k+2}!(\mu\mathfrak{d})^{n_{k+2}-1}}t^{n_k-1}\theta^{(n_{k+2}-n_k-2)_+}\delta^2\varepsilon^{\mathfrak{a}}\sum_{r=1}^{n_{k+2}}\sum_{\underline{\rho}\in\mathscr{P}^r_{n_{k+2}}}(\mathbf{r}-1)!\,.$$

The last sums can be bounded by

$$\frac{1}{n_{k+2}!}\sum_{r=1}^{n_{k+2}}\sum_{\underline{\rho}\in\mathscr{P}^r_{n_{k+2}}}(\mathbf{r}-1)! = \frac{1}{n_{k+2}!}\sum_{\mathbf{r}=1}^{n_{k+2}}\sum_{\substack{k_1+\cdots+k_{\mathbf{r}}=n_{k+2}\\ k_i\geq 1}}\frac{n_{k+2}!}{k_1!\cdots k_{\mathbf{r}}!}\frac{(\mathbf{r}-1)!}{\mathbf{r}!}$$
$$\leq \sum_{\mathbf{r}=1}^{n_{k+2}}\sum_{\substack{k_1+\cdots+k_{\mathbf{r}}=n_{k+2}\\ k_i\geq 1}}\frac{1}{k_1!\cdots k_{\mathbf{r}}!}\leq e^{n_{k+2}}$$

This ends the proof of the first inequality. □

Proof of (8.3). As the $\bar{\Phi}^{k'}_{\underline{n}_{k+2}}$ are symmetric, it is sufficient to study

$$\left|\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{[n_{k+2}]})\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{[n_{k+2}+1-m,2n_{k+2}-m]})\right|. \tag{8.12}$$

The bound (8.7) leads to

$$(8.12)\leq\frac{\|h\|^2}{(n_{k+2}!)^2}\sum_{\substack{(q_1,\omega_1\omega_2)\\(q_1'\omega_1'\omega_2')}}\sum_{\varpi\subset[n_{k+2}]}\sum_{\substack{(s_i)_i\\(s_i')_i}}\sum_{\substack{(\lambda_1,\cdots,\lambda_{\mathfrak{l}})\\(\lambda_1',\cdots,\lambda_{\mathfrak{l}}')}}\mathbb{1}_{\mathscr{R}^{0,t_s-\delta}_{\{q_1\},(s_i)_i}}(\bar{\mathsf{Z}}_{\omega_1}(\delta))\mathbb{1}_{\mathscr{R}^{0,t_s-\delta}_{\{q_1\},(s_i')_i}}(\bar{\mathsf{Z}}'_{\omega_2'}(\delta))$$
$$\times\left(\sum_{T\in\mathscr{T}([\mathfrak{l}])}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\Delta\!\!\!\Delta^{[n_{k+1}]}_{|\lambda_1|}(Z_{\lambda_1})\prod_{i=2}^{\mathfrak{l}}\Delta\!\!\!\Delta_{|\lambda_i|}(Z_{\lambda_i})\right)$$
$$\times\left(\sum_{T'\in\mathscr{T}([\mathfrak{l}'])}\prod_{(i,j)\in E(T')}\mathbb{1}_{\lambda'_i\overset{\circ}{\sim}\lambda'_j}\Delta\!\!\!\Delta^{[n_{k+1}]}_{|\lambda_1'|}(Z_{\lambda_1'})\prod_{i=2}^{\mathfrak{l}}\Delta\!\!\!\Delta_{|\lambda_i'|}(Z_{\lambda_i'})\right)\mathbb{1}_{\mathfrak{n}(k'\delta)=n_k}\mathbb{1}_{\mathscr{O}_\varpi}.$$

where we have denoted $\mathscr{T}(\sigma)$ the set of connected and simply connected graphs with vertices σ, a finite set. The sets $\mathscr{O}_\varpi$ have been defined in Definition 8.2.1. In addition,

- $q_1\in[n_{k+2}]$ and $q_1'\in[n_{k+2}+1-m,2n_{k+2}-m]$,
- $\omega_1\sqcup\omega_2=[n_{k+2}]$, $\omega_1'\sqcup\omega_2'=[n_{k+2}+1-m,2n_{k+2}-m]$, $q_1\in\omega_1$, $q_1\in\omega_1$ and $|\omega_1|=|\omega_1'|=n_{k+1}$,
- $\lambda_1\supset\omega_1$, $\lambda_1'\supset\omega_1'$, $(\lambda_2,\cdots,\lambda_{\mathfrak{l}})$ an unordered partition of $[n_{k+2}]\backslash\omega_1$ and $(\lambda_2',\cdots,\lambda_{\mathfrak{l}}')$ an unordered partition of $[n_{k+2}+1-m,2n_{k+2}-m]\setminus\omega_1'$.

The pseudotrajectory $\bar{Z}(\tau)$ (respectively $\bar{Z}'(\tau)$) begins with coordinates $Z_{[n_{k+2}]}$ (respectively with coordinates $Z_{[n_{k+2}+1-m,2n_{k+2}-m]}$) with parameters $(q_1,\omega_1,\omega_2,(\lambda_1,\cdots,\lambda_{\mathfrak{l}}))$ (respectively $(q_1',\omega_1',\omega_2',(\lambda_1',\cdots,\lambda_{\mathfrak{l}'}')))$ in the same way than in the proof of (8.2).

Note that the right hand-side is invariant under translation. Thus one can fix $x_1=0$ and integrate with respect to the other variables.

For a position $Z_{2n_{k+2}-m}$, we consider $\underline{\rho}:=(\rho_1,\cdots,\rho_{\mathbf{r}})$ the possible clusters. As in the previous section, with a little lost of symmetry, one can suppose that $\varpi_1\subset\rho_1$. We can then construct the parameters $\mathfrak{p}_1:=(\underline{\omega}^1,\underline{\omega}'^1,\underline{\lambda}^1,\underline{\lambda}'^1,\varpi)$, $(\mathfrak{p}_i)_{i\geq 2}:=\big((\underline{\omega}^i,\underline{\omega}'^i,\underline{\lambda}^i,\underline{\lambda}'^i)\big)_{i\geq 2}$:

- $\underline{\omega}^i:=(\omega_1^i,\omega_2^i)$ is a partition of $\rho_i\cap[n_{k+2}]$ defined by $\omega_j^i:=\omega_j\cap\rho_i$,

- $\underline{\omega'}^i := (\omega'^i_1, \omega'^i_3)$ is a partition of $\rho_i \cap [n_{k+2}+1-m, 2n_{k+2}+m]$ defined by $\omega'^i_j := \omega'_j \cap \rho_i$,
- $\underline{\lambda}^i := \{\lambda^i_1 := \lambda_1 \cap \rho_i\} \cup \{\lambda_j \text{ for } j \geq 2 \text{ with } \lambda_j \subset \rho_i\}$ a partition of $[n_{k+2}] \cap \rho_i$,
- $\underline{\lambda'}^i := \{\lambda'^i_1 := \lambda'_1 \cap \rho_i\} \cup \{\lambda'_j \text{ for } j \geq 2 \text{ with } \lambda'_j \subset \rho_i\}$ a partition of $[n_{k+2}+1-m, 2n_{k+2}+m] \cap \rho_i$.

We denote now $\mathfrak{P}(\rho_i)$ the new set of possible parameters $\mathfrak{p}_i$ (this will not create a conflict with the previous section). Because each cluster ρ_i is of size at most γ, $|\mathfrak{P}(\rho_i)|$ is bounded by some constant C_γ depending only on γ. Defining

$$\Delta_{\mathfrak{p}_1}(Z_{\rho_1}) := \mathbb{1}_{\substack{Z_{\rho_1} \text{ form a} \\ \text{possible cluster}}} \mathbb{1}_{\mathscr{O}_{\varpi}}, \quad \forall i \geq 2,\ \Delta_{\mathfrak{p}_i}(Z_{\rho_i}) := \mathbb{1}_{\substack{Z_{\rho_i} \text{ form a} \\ \text{possible cluster}}}$$

$$\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{(s_i)_i,(s'_i)_i} := \Big\{ Z_{2n_{k+2}-m} \in \mathscr{D}^{\underline{\rho}}_{\bar{\varepsilon}},\ \bar{\mathsf{Z}}_{\omega_1\cup\omega_2\cup\omega_3}(\delta) \in \mathscr{R}^{0,k\theta+(k-1)\delta}_{\omega_1,(s_i)_i},\ \bar{\mathsf{Z}}'_{n_{k+2}}(\delta) \in \mathscr{R}^{0,k\theta+(k-1)\delta}_{\omega'_1,(s'_i)_i} \Big\},$$

we have as in the inequality (8.8)

$$(8.12) \leq \frac{\gamma^{2(\gamma-2)n_{k+2}} \|h\|_0^2}{(n_{k+2}!)^2} \sum_{\mathbf{r}=1}^{2n_{k+2}-m} \sum_{\underline{\rho}\in\mathscr{P}^r_p} \sum_{\substack{(s_i)_i,(s'_i)_i \\ \underline{\mathfrak{p}}\in\prod_i \mathfrak{P}(\rho_i)}} \mathbb{1}_{\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{(s_i)_i,(s'_i)_i}}(Z_{2n_{k+2}-m}) \prod_{i=1}^{\mathbf{r}} \Delta_{\mathfrak{p}_i}(Z_{\rho_i}).$$

Note that, for at least one i, $\underline{\varpi}^i$ is not empty. We are now constructing a clustering tree in order to estimate $\mathscr{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{(s_i)_i,(s'_i)_i}$.

Consider the collision graph associated with the first pseudotrajectory $\mathscr{G}^{[0,t-t_s]}_{\omega_1\cup\omega_2}$ and the graph associated with the second one $\mathscr{G}^{[0,t-t_s]}_{\omega'_1\cup\omega'_2}$. Merge them and identify vertices in a same cluster ρ_i. Keeping only the first clustering collisions, we obtain the oriented tree $T^> := (\nu_i, \bar{\nu}_i)_{1\leq i\leq \mathbf{r}-1}$. Note that these clustering collisions can happen in the first or second pseudotrajectories.

As in the proof of (8.2) we have to bound the number of collisions of $T^>$ in the time interval $[0, 2\tau]$. There are at most $(n_k-1+n_{k+2}-m)$ collisions during $[(k'+1)\delta, t-t_s]$ (n_k-1 for the first pseudotrajectory, and we have to connect $n_{k+2}-m$ particles in the second). Thus, there are at least $(\mathbf{r}-(n_k-1+n_{k+2}-m))_+$ clustering collisions in $[\delta, (k'+1)\delta] \subset [0, 2\tau]$.

We explain quickly how to estimate the i-th collision. As in the previous paragraph, we construct the modified tree parameters $(\nu_{(i)}, \bar{\nu}_{(i)})$ and the change of variables

$$\forall i \in \{1, \cdots, \mathbf{r}-1\},\ \hat{x}_i := x_{\min \nu_{(i)}} - x_{\min \bar{\nu}_{(i)}},\ \tilde{X}_i := \mathrm{tr}_{x_{\min \rho_i}} X_{\rho_i},$$

$$X_{2,2n_{k+2}-m} \mapsto (\hat{x}_1 \cdots, \hat{x}_{\mathbf{r}-1}, \tilde{X}_1, \cdots, \tilde{X}_r),$$

and we integrate on the $(\hat{x}_i)$.

The clustering set B_i is defined as follows: fix t_{i+1} the time of the $(i+1)$-th clustering collision and the relative positions $\hat{x}_{i-1}, \cdots, \hat{x}_1$. We define the i-th clustering set

$$B_i := \bigcup_{\substack{q\in\bigcup_{j\in\nu_{(i)}}\rho_j \\ \bar{q}\in\bigcup_{\bar{j}\in\bar{\nu}_{(i)}}\rho_{\bar{j}}}} \Big(B^{q,\bar{q}}_i \cup B'^{q,\bar{q}}_i \Big)$$

with

$$B^{q,\bar{q}}_i := \Big\{ \hat{x}_i \ \Big|\ \exists \tau_i \in [0, t_{i+1} \wedge T_i],\ |\bar{\mathsf{x}}_{\bar{q}}(\tau_i) - \bar{\mathsf{x}}_q(\tau_i)| = \varepsilon \Big\},$$

where $T_i := 2\theta$ for the $(\mathbf{r}-n_k)_+$ first collisions, and t else. The set $B'^{q,q'}_i$ is defined in the same way for the other pseudotrajectory. We can apply the estimates of the previous paragraph:

$$\int \mathbb{1}_{B_i}\, \mathrm{d}\hat{x}_i \leq \frac{2C}{\mu\mathfrak{d}} \sum_{\substack{\nu_i\in\nu_{(i)} \\ \bar{\nu}_i\in\bar{\nu}_{(i)}}} \Big(|\rho_{\nu_i}| + \mathscr{H}_{|\rho_{\nu_i}|}(Z_{\rho_{\nu_i}})\Big)\Big(|\rho_{\bar{\nu}_i}| + \mathscr{H}_{|\rho_{\bar{\nu}_i}|}(Z_{\rho_{\bar{\nu}_i}})\Big) \int_0^{\tau_{i+1}\wedge T_i} \mathrm{d}\tau_i.$$

In this way, we end up with the same situation as in the estimate of (8.2), and we can apply the same strategy:

$$\int |\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{k+2})\bar{\Phi}^{k'}_{\underline{n}_{k+2}}(Z_{n_{k+2}+1-m,2n_{k+2}-m})| \frac{e^{-\mathscr{H}_{2_{k+2}-m}(Z_{2n_{k+2}-m})}}{(2\pi)^{2n_{k+2}-m}} \, \mathrm{d}X_{2,2n_{k+2}-m} \, \mathrm{d}V_{2n_{k+2}-m}$$
$$\leq \frac{(2n_{k+2}-m)!\|h\|^2}{(n_{k+2}!)^2(\mu\mathfrak{d})^{2n_{k+2}-m-1}} C^{n_{k+2}}\delta^2\varepsilon^{\mathfrak{a}}\tau^{(n_{k+2}-n_k-2)_+}t^{n_k-1+n_{k+2}-m}$$
$$\leq \frac{\mu^{m-1}}{n_{k+2}^m}\left(\frac{\|h\|}{(\mu\mathfrak{d})^{n_{k+2}-1}}\tilde{C}^{n_{k+2}}\right)^2 \delta^2\varepsilon^{\mathfrak{a}}\theta^{(n_{k+2}-n_k-2)_+}t^{n_k-1+n_{k+2}}$$

which concludes the proof. □

Appendix A. The Linearized Boltzmann operator without cut-off

In this section, we construct the linearized Boltzmann operator associated with the power law $1/r^s$, $s > 1$ and we explain where the scaling $\mathfrak{d}_{s,\alpha} = \alpha^{2/s}$ comes from.

We begin with a change of variables in the definition of the Boltzmann operator $\mathscr{L}_\alpha$. For (v, v_*, ν) we define

$$\vec{\rho} := \tfrac{\nu\wedge(v-v_*)}{|v-v_*|} \in \mathrm{span}(v-v_*)^\perp \tag{A.1}$$

the *impact parameters*, with the Jacobian

$$((v-v_*)\cdot\nu)_+ \, \mathrm{d}\nu \to |v-v_*| \, \mathrm{d}\vec{\rho}.$$

This allows us to redefine the post-collisional velocities (v', v_*) for an interaction potential $\mathscr{U}$

$$\begin{cases} (v', v'_*) := \lim_{t\to\infty}(v_a(t), v_b(t)) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(x_a, x_b) = (v_a, v_b), \ \dfrac{\mathrm{d}}{\mathrm{d}t}(v_a, v_b) = \alpha(-\nabla\mathscr{U}(x_b-x_a), \nabla\mathscr{U}(x_b-x_a)) \\ \lim_{t\to-\infty}(v_a(t), v_b(t)) =: (v, v_*), \ (v_a-v_b)\wedge(x_a-x_b) = |v-v_*|\vec{\rho}. \end{cases} \tag{A.2}$$

With this definition, the scattering map can easily be defined for a not compactly supported decreasing potential.

For $\lambda > 0$, we make the change of coordinate

$$(t, x_a, x_b, v_a, v_b) \mapsto (\tilde{t}, \tilde{x}_a, \tilde{x}_b, v_a, v_b) := (\lambda t, \lambda x_a, \lambda x_b, v_a, v_b)$$

In the new coordinates, the equations of motion become

$$\tfrac{\mathrm{d}}{\mathrm{d}\tilde{t}}(\tilde{x}_a, \tilde{x}_b) = (\tilde{v}_a, \tilde{v}_b), \ \tfrac{\mathrm{d}}{\mathrm{d}\tilde{t}}(\tilde{v}_a, \tilde{v}_b) = \alpha\left(-\nabla\mathscr{U}\left(\tfrac{\tilde{x}_a-\tilde{x}_b}{\lambda}\right), \nabla\mathscr{U}\left(\tfrac{\tilde{x}_a-\tilde{x}_b}{\lambda}\right)\right)$$

Hence, the post-collisional parameters associated with $(v, v_*, \vec{\rho})$ and potential $\mathscr{U}$ are the same than the ones associated to $(v, v_*, \lambda\vec{\rho})$ and potential $\mathscr{U}(\cdot/\lambda)$.

Performing the change of variable $\vec{\rho} \to \alpha^{-1/s}\vec{\rho}$ in the collisional operator gives

$$\alpha^{2/s}\mathscr{L}_{\alpha\mathscr{V}}g = \mathscr{L}_{\mathscr{U}_\alpha}g, \text{ where } \mathscr{U}_\alpha(r) := \alpha\mathscr{V}(r\alpha^{1/s}) = \frac{f\left(r\alpha^{\frac{1}{s}}\right)}{r^s}.$$

This new potential converges when $\alpha \to 0$ to $\mathscr{U}^s(r) := 1/r^s$. It is natural to guess the convergence of the operators

$$\frac{1}{\alpha^{2/s}}\mathscr{L}_\alpha \to \mathscr{L}_{\mathscr{U}^s}.$$

Appendix B. Geometrical estimates

B.1. **Estimation of the length time of a collision.**

Lemma B.1. *Let $\mathscr{V}$ an interaction potential which is radial and supported in a ball of diameter ε. We consider two particles, 1 and 2, with initial coordinates*

$$(x_a(0), v_a(0)) = (0, v_1), \ (x_b(0), v_b(0)) = (\varepsilon\nu, v_2), \ \nu\in\mathbb{S}^{d-1}, \ (v_1-v_2)\cdot\nu < 0,$$

following the Hamiltonian dynamics linked to

$$H := \frac{|v_a|^2+|v_b|^2}{2} + \mathscr{V}(x_a-x_b).$$

Then the collision time is bound by

$$T := \inf\{\tau > 0 \; |x_a(\tau) - x_b(\tau)| > \varepsilon\} \leq \frac{\varepsilon}{|v_1 - v_2|}. \tag{B.1}$$

Proof. The motion equations can be written as

$$\begin{aligned} \tfrac{d}{dt}(x_a + x_b) &= (v_a + v_b) \quad & \tfrac{\mathrm{d}}{\mathrm{d}t}(v_a + v_b) &= 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t}(x_a - x_b) &= (v_a - v_b) \quad & \tfrac{\mathrm{d}}{\mathrm{d}t}(v_a - v_b) &= -2\nabla\mathscr{V}(x_a - x_b)\,. \end{aligned}$$

Hence, T does not depend on $v_1 + v_2$.

We use the impact parameter $\rho := |\vec{\rho}|$ defined in (A.1). The time T_α is given by (See chapter 8 of [GSRT13])

$$T = \frac{2}{|V|}\int_{r_{\min}}^{\varepsilon/2} \frac{\mathrm{d}r}{\sqrt{1 - \frac{\rho^2}{r^2} - 2\frac{\mathscr{V}(r)}{|v_1 - v_2|^2}}}, \tag{B.2}$$

with $r_{\min}$ defined by

$$1 - \frac{\rho^2}{r_{\min}^2} - 2\frac{\mathscr{V}(r_{\min})}{|v_1 - v_2|^2} = 0.$$

We begin by performing the change of variables

$$u^2 := \frac{\rho^2}{r^2} + 2\frac{\mathscr{V}(r)}{|v_1 - v_2|^2} \tag{B.3}$$

which implies

$$T = \frac{2}{|V|}\int_{\rho/\varepsilon}^{1} \frac{u\,\mathrm{d}u}{\sqrt{1-u^2}} \frac{r}{\frac{\rho^2}{r^2} - \frac{r\mathscr{V}'(r)}{|v_1 - v_2|^2}} \tag{B.4}$$

Using that $\mathscr{V}'$ is non positive and that $u \leq \frac{\rho}{r}$,

$$T \leq \frac{2}{|v_1 - v_2|}\int_{\rho/\varepsilon}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}}\frac{ur^3}{\rho^2} \leq \frac{2}{|v_1 - v_2|}\int_{\rho/\varepsilon}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}}\frac{ur^3}{\rho^2} \leq \frac{2}{|v_1 - v_2|v}\int_{\rho/\varepsilon}^{1} \frac{\rho\,\mathrm{d}u}{u^2\sqrt{1-u^2}}$$

The variables are turned into $x = u/\rho$,

$$T \leq \frac{2}{|v_1 - v_2|}\int_{\rho}^{\varepsilon} \frac{x\,\mathrm{d}x}{\sqrt{x^2 - \rho^2}} = \frac{2\sqrt{\varepsilon^2 - \rho^2}}{|v_1 - v_2|} \leq \frac{2\varepsilon}{|v_1 - v_2|}.$$

□

B.2. **Proof of Proposition 7.2.** The goal of this section is to prove the following estimation 7.5:

Lemma B.2. *Let $\mathscr{V}$ an interaction potential that is radial, decreasing, and supported in a ball of radius ε.*

There > 0 independent of $\mathscr{V}$ such that for $t = n_k\theta$

$$\sum_{T=(q_i,\bar{q}_i,s_i)_{i\leq n_k-1}} \int_{\mathfrak{T}_{n_k}\times\mathfrak{G}_T^0} \left(1 - \mathbb{1}_{\mathfrak{G}_T^0}\right)\Lambda_T(V_{n_K}, \nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_i M^{\otimes n_k} dV_{n_k} \tag{B.5}$$
$$\leq C^{n_K}(n_k)^{n_k}\theta^{(n_k - n_{k-1}-2)_+}t^{n_k}\varepsilon^{1/4}.$$

The proof of this lemma is an adaptation of the proof of Lemma 8 of [PSS14]. The estimation is not optimal. For example, the factor $\varepsilon^{1/4}$ can be replaced in the hard spheres setting by $\varepsilon|\log\varepsilon|^r$ for some constant $r > 1$ (see for example [BGSRS21]). However, optimal estimates use the upper bound of the collision kernel of hard spheres. Such bounds are verified for more general potential and certainly not in the limit $\alpha \to 0$. The proof of [PSS14] (which is adapted from it) is more robust.

Proof. We need to avoid

- an *overlap*: there exists a time $\tau \in (0,t) \cap \delta\mathbb{Z}$ and two particles q and q' such that
$$|\mathsf{x}_q(\tau) - \mathsf{x}_{q'}(\tau)| \leq \varepsilon,$$
- a *recollision*: there exists a time $\tau \in [0,t]$ and two particles q and q' such that $\tau \notin \{\tau_1, \cdots, \tau_{n_k-1}\}$ and
$$|\mathsf{x}_q(\tau) - \mathsf{x}_{q'}(\tau)| = \varepsilon \text{ and } (\mathsf{x}_q(\tau) - \mathsf{x}_{q'}(\tau))\cdot(\mathsf{v}_q(\tau) - \mathsf{v}_{q'}(\tau)) < 0.$$

We begin with the estimation of overlap, which is easier.

As the i-th collision between particles $(q_i,\bar{q}_i)$ can last only a time $\frac{\varepsilon}{|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)|}$ (B.1), there can be an overlap only if there is some $\tau\in\delta\mathbb{Z}\cap[0,t]$ such that τ_i is in the interval

$$I_\varepsilon(\tau,V_{n_K},\nu_{[n_k-1]}):=\left[\tau-\frac{\varepsilon}{|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)|},\tau\right]$$

Hence, the set of parameters leading to an overlap is smaller than

$$\begin{aligned}\sum_{\substack{\tau\in\delta\mathbb{Z}\cap[0,t]\\1\le i\le n_k-1}}\sum_T\int_{\mathfrak{T}_{n_k}\times\mathfrak{G}_T^0}&\mathbb{1}_{I_\varepsilon(\tau,V_{n_K},\nu_{[n_k-1]})}(\tau_i)\Lambda_T(V_{n_K},\nu_{[n_k-1]})\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tau_{[n_k-1]}M^{\otimes n_k}dV_{n_k}\\&\le\frac{t}{\delta}\frac{C^{n_k}t^{n_{k-1}}\theta^{(n_k-n_{k-1}-1)_+}}{n_k^{n_k}}\sum_{1\le i\le n_k-1}\sum_T\int_{\mathfrak{G}_T^0}\frac{\varepsilon\Lambda_T(V_{n_K},\nu_{[n_k-1]})}{|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)|}\,\mathrm{d}\nu_{[n_k-1]}M^{\otimes n_k}dV_{n_k}\end{aligned}$$

As

$$\frac{\varepsilon\Lambda_T(V_{n_K},\nu_{[n_k-1]})}{|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)|}=\prod_{\substack{j=1\\j\neq i}}^{n_k-1}\left(\left(\mathsf{v}^\varepsilon_{q_j}(\tau_i)-\mathsf{v}^\varepsilon_{\bar{q}_j}(\tau_j)\right)\cdot\nu_j\right)_+$$

we can apply the same estimates as in Lemma 6.1. We conclude that the set of overlap has a measure smaller than

$$\frac{\varepsilon C^{n_k}t^{n_{k-1}+1}\theta^{(n_k-n_{k-1}-1)_+}}{\delta n_k^{n_k}}.$$

We treat the recollision now. If the first recollision involves particles q and q' at time τ_{rec}, we consider $\omega\subset[n_{k+2}]$ the connected components of $\{q,q'\}$ in the collision graph $\mathscr{G}^{[0,\tau_{\mathrm{rec}})}$. Before the time τ_{rec}, the pseudotrajectory $\mathsf{Z}^\varepsilon_\omega(\tau)$ and its formal limit $\mathsf{Z}_\omega(\tau)$ are close up to a translation, and using Lemma 6.3, there exists a $y_0\in\mathbb{T}$ such that

$$\forall\tau\in[0,\tau_{\mathrm{rec}}],\ |\mathsf{X}^0_\omega(\tau)-\mathrm{tr}_{y_0}\mathsf{X}^\varepsilon_\omega(\tau)|\le\sum_{i=1}^{n_{k+2}-1}\frac{2n_K\mathbb{V}\varepsilon}{\left|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)\right|}.$$

Hence, there is a recollision if at time $\tau_{\mathrm{rec}}\in[0,t]\setminus\{\tau_i\}_i$,

$$\exists\,q,q'\in[n_k-1]\text{ such that }|\mathsf{x}^0_q(\tau_{\mathrm{rec}})-\mathsf{x}^0_{q'}(\tau_{\mathrm{rec}})|\le\varepsilon+\sum_{i=1}^{n_{k+2}-1}\frac{2n_K\mathbb{V}\varepsilon}{\left|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{\bar{q}_i}(\tau_i^-)\right|}.\tag{B.6}$$

We can study only the limiting flow and defining an touch as "there exists a time τ_{rec} such that (B.6) is verified: we have

$$\mathfrak{T}_{\underline{n}_k}\times\mathfrak{G}_T^0\setminus\mathfrak{T}_{\underline{n}_k}\times\mathfrak{G}_T^\varepsilon\subset\mathfrak{T}_{\underline{n}_k}\times\mathfrak{G}_T^0\cap\{\text{at least one touch}\}.$$

The first step is to forbid the collisions which last to long. We define $\mathscr{E}_1\subset\mathfrak{T}_{\underline{n}_k}\times\mathfrak{G}_T^0$ as

$$\mathscr{E}_1:=\left\{\forall i\le n_k-1,\left|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{q_i}(\tau_i^-)\right|\min\{1,(\tau_i-\tau_{i-1}),(\tau_{i+1}-\tau_i)\}\ge\frac{\varepsilon^{1/4}}{n_k^2\mathbb{V}}\right\}.$$

In $\mathscr{E}_1$ there is a touch if there exists a time τ_{rec} such that

$$|\mathsf{x}^0_q(\tau_{\mathrm{rec}})-\mathsf{x}^0_{q'}(\tau_{\mathrm{rec}})|\le3\varepsilon^{3/4},\tag{B.7}$$

and

$$1-\mathbb{1}_{\mathscr{E}_1}\le\sum_{i=1}^{n_k-1}\mathbb{1}_{\left|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{q_i}(\tau_i^-)\right|\le\frac{\varepsilon^{c_1}}{n_k^2\mathbb{V}}}+\mathbb{1}_{\left|\mathsf{v}_{q_i}(\tau_i^-)-\mathsf{v}_{q_i}(\tau_i^-)\right|\max\{(\tau_i-\tau_{i-1}),(\tau_{i+1}-\tau_i)\}\le\frac{\varepsilon^{1/4}}{n_k^2\mathbb{V}}}.$$

Now fix a collision tree T. The first touch happens at time τ_{rec} between particles q_{rec} and q'_{rec}. There exists a collision i_0, two disjoint sequences of collisions $(i_j)_{j\le p}$ and $(i'_j)_{j\le p'}$ and two sequences of particles $(a_j)_{j\le p}$, $(a'_j)_{j\le p'}$ such that

- $\forall j\ge1$, $i_0<i_j$ and $i_0<i'_j$,
- $\forall j\ge1$, $a_j\in\{q_{i_j},q'_{i_j}\}\cap\{q_{i_{j-1}},q'_{i_{j-1}}\}$ and $a'_j\in\{q_{i'_j},q'_{i'_j}\}\cap\{q_{i'_{j-1}},q'_{i'_{j-1}}\}$,
- if for $j<j'$, $a_j=a_{j'}$, then for any $i\in[i_j,i_{j'}]$ such that $a_j\in\{q_i,q'_i\}$ we have $j\in\{i_j,i_{j+1},\cdots,i_{j'}\}$, and similary for the sequences $(i'_j)_{j\le p'}$, $(a'_j)_{j\le p'}$,
- $a_0=q_{i_0}$, $a'_0=q'_{i_0}$ and $\{a_p,a'_{p'}\}=\{q_{\mathrm{rec}},q'_{\mathrm{rec}}\}$.

We have a touch if

$$\min_{\substack{s\in[-t,t]R\\ y_0\in\mathbb{Z}^d}} \Big| y_0 + \sum_{j=1}^{p} \mathsf{v}_{a_j}(]\tau_{i_{j-1}},\tau_{i_j}[)(\tau_{i_j}-\tau_{i_{j-1}}) + \mathsf{v}_{a_p}(]\tau_{i_p},\tau_{\mathrm{rec}}[)(s-\tau_{i_p}) - \sum_{j=1}^{p'} \mathsf{v}_{a'_j}(]\tau_{i'_{j-1}},\tau_{i'_j}[)(\tau_{i'_j}-\tau_{i'_{j-1}}) + \mathsf{v}_{a'_{p'}}(]\tau_{i'_{p'}},\tau_{\mathrm{rec}}[)(s-\tau_{i'_{p'}}) \Big| \leq \varepsilon^{3/4}. \tag{B.8}$$

As the velocities are bound by $n_k\mathbb{V}$, y_0 is smaller than $n_k^2\mathbb{V}t$.

Remark B.2.1. *Note that we can perform the previous construction if $\tau_{\mathrm{rec}} > \tau_{i_0}$. If we study the overlap for particles of size ε, it is always the case. But for punctual particles, the pathology can happen before the first collision. Our proof can be adapted by taking i_0 such that $i_0 > i_j$ and $i_0 > i')_j$, making the change of variable $V_{n_k} \mapsto \mathsf{V}_{n_k}(\tau_{i_0}^-)$ and looking everything backwardly.*

Denoting

$$\Delta\mathsf{v}_j := \mathsf{v}_{a_{j+1}}(]\tau_{i_j},\tau_{i_{j+1}}[) - \mathsf{v}_{a_j}(]\tau_{i_{j-1}},\tau_{i_j}[),\ \Delta\mathsf{v}'_j := \mathsf{v}_{a'_{j+1}}(]\tau_{i'_j},\tau_{i'_{j+1}}[) - \mathsf{v}_{a'_j}(]\tau_{i'_{j-1}},\tau_{i'_j}[),$$
$$w_0 := \mathsf{v}_{a_0}(\tau_{i_0}^+) - \mathsf{v}_{a'_0}(\tau_{i_0}^+),\ \text{and}\ w_{\mathrm{f}} = \mathsf{v}_{a_p}(]\tau_{i_p},\tau_{\mathrm{rec}}[) - \mathsf{v}_{a'_{p'}}(]\tau_{i'_{p'}},\tau_{\mathrm{rec}}[),$$

the equation (B.8) can be written as

$$\min_{\substack{s\in[-t,t]\\ y_0\in\mathbb{Z}^d}} \Big| y_0 - \tau_{i_0}w_0 + \sum_{j=1}^{p}\Delta\mathsf{v}_j\tau_{i_j} - \sum_{j=1}^{p'}\Delta\mathsf{v}'_j\tau_{i'_j} + sw_{\mathrm{f}}\Big| \leq \varepsilon^{3/4} \tag{B.9}$$

We define $\mathscr{E}_2 \subset \mathscr{E}_1$ as

$$\mathscr{E}_2 := \Big\{\forall s\in[-t,t],\ i\in[1,n_k],\ y_0\in\mathbb{Z}^d\setminus\{0,\ \big|y_0 - s\left(\mathsf{v}_{q_i}(\tau_i^+) - \mathsf{v}_{q'_i}(\tau_i^+)\right)\big| \geq \varepsilon^{1/4}\Big\}. \tag{B.10}$$

We want to bound the complement of $\mathscr{E}_2$ in $\mathscr{E}_1$. As it is a subset of $\mathscr{E}_1$, the case $y_0 = 0$ is impossible. The velocities are bounded by $\mathbb{V}$ and we only have to test the y_0 such that $|y_0| \leq t\mathbb{V}$. If

$$\big|y_0 - s\left(\mathsf{v}_{q_i}(\tau_i^+) - \mathsf{v}_{q'_i}(\tau_i^+)\right)\big| \leq \varepsilon^{1/4},$$

the vector $\left(\mathsf{v}_{q_i}(\tau_i^+) - \mathsf{v}_{q'_i}(\tau_i^+)\right)$ has to leave in a cone $\mathscr{C}(y_0,\varepsilon^{1/4})$ of axes y_0 and angle $2\arcsin\varepsilon^{1/4}/|y_0| \lesssim 2\varepsilon^{1/4}$ (y_0 is of length at least 1). Hence

$$\mathbb{1}_{\mathrm{touch}}\mathbb{1}_{\mathscr{E}_1}(1-\mathbb{1}_{\mathscr{E}_2}) \leq \sum_{\substack{y_0\in\mathbb{Z}^d\\ |y_0|\leq t\mathbb{V}}}\sum_{i=1}^{n_k-1}\mathbb{1}_{\mathsf{v}_{q_i}(\tau_i^+)-\mathsf{v}_{q'_i}(\tau_i^+)\in\mathscr{C}(y_0,\varepsilon^{1/4})}. \tag{B.11}$$

Now we place our-self in $\mathscr{E}_2$. There exists a ℓ_0 such that $|\Delta\mathsf{v}_{\ell_0}|$ or $|\Delta\mathsf{v}'_{\ell_0}|$ is smaller than $\frac{\varepsilon^{1/4}}{2n_kt}$ (in the following, we suppose that it is $|\Delta\mathsf{v}_{\ell_0}|$). Else, by triangular inequality

$$|w_f - w_0| = \Big|\sum_{j=1}^{p}\Delta\mathsf{v}_j - \sum_{j=1}^{p'}\Delta\mathsf{v}'_j\Big| \leq \frac{\varepsilon^{1/4}}{3n_kt},$$

which gives the following contradiction

$$\exists s\in\mathbb{R},\ |y_0 - (\tau_{i_0}-s)w_0| \leq \varepsilon^{3/4} + \frac{2\varepsilon^{1/4}}{3}.$$

In the following, we denote $\hat{w}_{\mathrm{f}} := \frac{w_{\mathrm{f}}}{|w_{\mathrm{f}}|}$. Equation (B.9) implies that

$$\min_{y_0\in\mathbb{Z}^d}\Big|(y_0-\tau_{i_0}w_0)\wedge\hat{w}_{\mathrm{f}} + \sum_{j=1}^{p}\Delta\mathsf{v}_j\wedge\hat{w}_{\mathrm{f}}\tau_{i_j} - \sum_{j=1}^{p'}\Delta\mathsf{v}'_j\wedge\hat{w}_{\mathrm{f}}\tau_{i'_j}\Big| \leq \varepsilon^{3/4}. \tag{B.12}$$

We distinguish two cases:

- If there exists a $\ell_1\in\mathbb{N}$ such that $|\Delta\mathsf{v}_{\ell_1}\wedge\hat{w}_{\mathrm{f}}| \geq \varepsilon^{1/2}/n_k$, $\tau_{i_{\ell_1}}$ has to be in an interval of length $\varepsilon^{1/4}$.
- Else, we have

$$|w_0\wedge\hat{w}_{\mathrm{f}}| \leq \sum_{j=1}^{p}|\Delta\mathsf{v}_j\wedge\hat{w}_{\mathrm{f}}| + \sum_{j=1}^{p'}|\Delta\mathsf{v}_j\wedge\hat{w}_{\mathrm{f}}| \leq \varepsilon^{1/2}.$$

If y_0 is non zero, $|y_0 \wedge w_f|$ is smaller than $2t\mathbb{V}\varepsilon^{1/2}$. Hence, there exists a collision j, a vector $y_0 \in \mathbb{Z}^d$ with $|y_0| \leq n_K t\mathbb{V}$, and a couple of particles (q,q') such that

$$\left|\left(\mathsf{v}_q(\tau_j^-) - \mathsf{v}_{q'}(\tau_j^-)\right) \wedge y_0\right| \leq 2t\mathbb{V}\varepsilon^{1/2}$$

If $y_0 = 0$, as we are in $\mathscr{E}_1$,

$$\left|\frac{\Delta\mathsf{v}_{\ell_0}}{|\Delta\mathsf{v}_{\ell_0}|} \wedge \hat{w}_{\mathrm{f}}\right| \leq \frac{\varepsilon^{1/2}}{n_k|\Delta\mathsf{v}_{\ell_0}|} \leq \varepsilon^{1/4} \text{ and } |w_0 \wedge \hat{w}_f| \leq \frac{1}{\tau_0}\left(\varepsilon^{3/4} + n_k\frac{\varepsilon^{1/2}}{n_k}\right) \leq \varepsilon^1/2.$$

Finally

$$\left|\frac{\Delta\mathsf{v}_{\ell_0}}{|\Delta\mathsf{v}_{\ell_0}|} \wedge w_0\right| \leq \varepsilon^{1/2} + \varepsilon^{1/4}. \tag{B.13}$$

Note that $\Delta\mathsf{v}_{\ell_0}$ is equal one of the $(\pm\zeta_i(\mathsf{v}_{q_{\ell_0}}(\tau_{\ell_0}^-) - \mathsf{v}_{q'_{\ell_0}}(\tau_{\ell_0}^-), \nu_{\ell_0})_{i\leq 4}$, where, denoting

$$(\tfrac{w'}{2}, -\tfrac{w'}{2}, \nu') = \xi_\alpha((w/2, -w/2, \nu)),$$

$$\zeta_1(w,\nu) := w,\ \zeta_2(w,\nu) := w',\ \zeta_3(w,\nu) := \frac{w-w'}{2} \text{ and } \zeta_4(w,\nu) := \frac{w+w'}{2}.$$

We conclude that

$$\begin{aligned}\int_{\mathfrak{T}_{n_k}} & \mathrm{d}\tau_{n_k} \mathbb{1}_{\mathrm{touch}} \\ \leq \frac{C^{n_k} t^{n_{k-1}} \theta^{(n_k - n_{k-1} - 1)_+}}{n_k^{n_k}} & \Bigg(\varepsilon^{1/4} + \sum_i \mathbb{1}_{|\mathsf{v}_{q_i}(\tau_i^-) - \mathsf{v}_{q_i}(\tau_i^-)| \leq \varepsilon^{1/4}} + \frac{\varepsilon^{1/4}}{|\mathsf{v}_{q_i}(\tau_i^-) - \mathsf{v}_{q_i}(\tau_i^-)|} \\ & + \sum_{\substack{y_0 \in \mathbb{Z}^d \\ |y_0| \leq tn_k\mathbb{V}}} \sum_{\substack{i \leq n_k \\ (q,q')}} \mathbb{1}_{\left|\left(\mathsf{v}_q(\tau_j^-) - \mathsf{v}_{q'}(\tau_j^-)\right)\wedge y_0\right| \leq 2t\mathbb{V}\varepsilon^{1/2}} \\ & + \sum_{i,j}\sum_{\ell=1}^4 \mathbb{1}_{\left|\frac{\zeta_\ell(\mathsf{v}_{q_j}(\tau_j^-) - \mathsf{v}_{q'_j}(\tau_j^-), \nu_j)}{|\zeta_\ell(\mathsf{v}_{q_j}(\tau_j^-) - \mathsf{v}_{q'_j}(\tau_j^-), \nu_j)|} \wedge (\mathsf{v}_{q_i}(\tau_i^+) - \mathsf{v}_{q'_i}(\tau_i^+))\right| \leq \varepsilon^{1/4}}\Bigg).\end{aligned} \tag{B.14}$$

We have to integrate now with respect to $(\nu_{[n_k-1]}, V_{n_k})$. As in the proof of (6.11) we use the applications Ξ_T^i defined by (6.13).

We recall that

$$\Xi_T^{i+1}\Xi_T^i \cdots \Xi_T^1(\nu_{[n_k-1]}, V_{n_k}) = (\tilde{\nu}_{[n_k-1]}, \tilde{V}_{n_k} = \mathsf{V}_{n_k}(\tau_i^+))$$

and that the Jacobian of this application is 1. We can integrate with respect to $\mathsf{v}_q(\tau_i^-) - \mathsf{v}_q(\tau_i^-)$.

We treat now the last singularity. If we remove the edges (q_j, q_j') from T, either q_j or q_j' is not in the connected component of $\{q_i, q_i'\}$. Without loss of generality, we suppose that it is q_j. We denote ω the connected components of q_j in $T \setminus \{(q_j, q_j', s_j)\}$. Before the collision j the particles of ω are independent of the other ones, and as before, we can construct an application of Jacobian 1

$$\bar{\Xi} : (\nu_{[n_k-1]}, V_{n_k}) \mapsto (\bar{\nu}, \mathsf{V}_{\omega^c})(\tau_i^+), \mathsf{V}_\omega(\tau_j^-)).$$

In addition

$$\begin{aligned}|V_{n_k}|^2 &= \frac{|\mathsf{V}_{n_k}(\tau_i^+)|^2}{2} + \frac{|\mathsf{V}_{n_k}(\tau_j^-)|^2}{2} \\ &\geq \frac{|\mathsf{V}_{\omega^c}(\tau_i^+)|^2}{2} + \frac{|\mathsf{v}_{q_j}(\tau_j^-)|^2 + |\mathsf{v}_{q_j}(\tau_j^-)|^2}{2} + \frac{|\mathsf{V}_{\omega\setminus\{q_j\}}(\tau_j^-)|^2}{2} \\ &\geq \frac{|\mathsf{V}_{\omega^c}(\tau_i^+)|^2}{2} + \frac{|\mathsf{v}_{q_j}(\tau_j^-) - \mathsf{v}_{q_j}(\tau_j^-)|^2}{4} + \frac{|\mathsf{V}_{\omega\setminus\{q_j\}}(\tau_j^-)|^2}{2},\end{aligned}$$

and denoting $w := \mathsf{v}_{q_j}(\tau_j^-) - \mathsf{v}_{q_j}(\tau_j^-)$, we can integrate with respect to the velocities

$$\begin{aligned}\sum_T \int_{\mathfrak{G}_T^0} \mathbb{1}_{\left|\frac{\zeta_\ell(\mathsf{v}_{q_j}(\tau_j^-)-\mathsf{v}_{q'_j}(\tau_j^-),\nu_j)}{|\zeta_\ell(\mathsf{v}_{q_j}(\tau_j^-)-\mathsf{v}_{q'_j}(\tau_j^-),\nu_j)|}\wedge(\mathsf{v}_{q_i}(\tau_i^+)-\mathsf{v}_{q'_i}(\tau_i^+))\right|\leq\varepsilon^{1/4}} \Lambda_T(V_{n_K},\nu_{[n_K-1]})M^{\otimes n_k}\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}V_{n_k} \\ \leq \int \mathbb{1}_{\left|\frac{\zeta_\ell(w,\nu_j)}{|\zeta_\ell(w,\nu_j)|}\wedge(v_{q_i}-v_{q'_i})\right|\leq\varepsilon^{1/4}} \frac{(C(1+|V_{n_k}|^2))^{n_k}e^{-\frac{|V_{[n_k]\setminus\{q_j\}}|^2}{4}-\frac{|w|^2}{8}}}{(2\pi)^{\frac{n_kd}{2}}}\,\mathrm{d}\nu_{[n_k-1]}\,\mathrm{d}\tilde{V}_{[n_k]}\,\mathrm{d}w \\ \leq n_k^{2n_k}C^{n_k}\varepsilon^{1/4}.\end{aligned}$$

Finally we obtain

$$\frac{1}{(\mu\mathfrak{d})^{n_k-1}}\sum_T\int_{\mathfrak{T}_{n_k}\times\mathfrak{G}_T^0}\mathbb{1}_{\mathrm{reco}}\prod_{i=1}^{n_k-1}|(\mathsf{v}_{q_i}(\tau_i)-\mathsf{v}_{q'_i}(\tau_i))\cdot\nu_i|\,\mathrm{d}\nu_i\,\mathrm{d}\tau_i M^{\otimes n_k}dV_{n_k} \leq \frac{(n_kC)^{n_k}t^{n_{k-1}}\theta^{(n_k-n_{k-1}-1)_+}}{(\mu\mathfrak{d})^{n_k-1}}\varepsilon^{1/4}. \tag{B.15}$$

This conclude the proof. □

B.3. **Proof of** (3.6)**.**

Lemma B.3. *For $n \leq 2\gamma$,*

$$\int \mathbb{1}_{\substack{Z_n\, form\, a\\ \text{possible cluster}}} M^{\otimes n}\,\mathrm{d}Z_n \leq C_\gamma \mu^{-n+1}\delta^{n-1} \tag{B.16}$$

Proof. First choose a family $\omega_1, \cdots, \omega_p$ of subset covering $[r]$, and $(\underline{\lambda}_i)_{i\leq p} = (\lambda_i^1, \cdots, \lambda_i^{l_i})_{i\leq p}$ a family of partitions of ω_i. As n is bounded, there are a finite number of possible $((\omega_i)_i, (\underline{\lambda}_i)_i)$. We construct the graph $\mathscr{G}$ as the merge of the collision graph of $\mathsf{Z}(\tau, Z_{\omega_i}, \underline{\lambda}_i)$ on $[0,\delta]$, and we extract T the clustering tree (there are less than $(2\gamma)^{2\gamma}$ possible clusterings trees). We can then adapt the proof of (5.3) (where we treated only two pseudotrajectories), and we obtain the expected result. □

B.4. **Proof of** (8.10)**.** We recall that $\mathscr{O}_r \subset \mathbb{D}^r$ is the set

$$\mathscr{O}_r := \Big\{ Z_r \in \mathbb{D}^r,\ \exists(\lambda_1,\cdots,\lambda_{\mathfrak{l}}),\ \text{the collision graph of } \mathsf{Z}_r(\cdot, Z_z, (\lambda_1,\cdots,\lambda_{\mathfrak{l}})) \text{ on } [0,\delta] \text{ is}$$
$$\text{connected and the pseudotrajectory has a collision or a multiple interaction}\Big\}.$$

Proposition B.4. *There exists a positive constant C_n depending only on the dimension and the number of particles n such that*

$$\int_{\mathbb{T}^{n-1}\times B_n(\mathbb{V})} \mathbb{1}_{\mathscr{O}_n}(Z_n)\frac{e^{-\mathscr{H}_r}}{(2\pi)^{\frac{nd}{2}}}\,\mathrm{d}X_{2,n}\,\mathrm{d}V_n \leq \frac{C_n}{(\mu\mathfrak{d})^{n-1}}\delta^{n-2}\varepsilon^{1/4} \leq \frac{C_n}{(\mu\mathfrak{d})^{n-1}}\delta^{n-1}\varepsilon^{1/12}, \tag{B.17}$$

where $B_r(\mathbb{V})$ is the ball of radius $\mathbb{V}$ in dimension rd (we use that $\delta = \varepsilon^{1/12}$).

Proof. For $r = 2$, we have

$$\int_{\mathbb{D}^2}\mathbb{1}_{\mathscr{O}_2}(Z_2)\frac{e^{-\mathscr{H}_2}}{(2\pi)^d}\,\mathrm{d}x_2\,\mathrm{d}V_2 = C\varepsilon^d \leq \frac{C\delta^2\varepsilon^{\mathfrak{a}}}{\mu\mathfrak{d}}$$

as $\delta^2\varepsilon^{\mathfrak{a}} = \varepsilon^{3/12} \leq \varepsilon$.

Fix parameters $(\lambda_1, \cdots \lambda_{\mathfrak{l}})$ and denote $\mathscr{R}_{(\lambda_1,\cdots\lambda_{\mathfrak{l}})} \subset \mathbb{D}^r$ the set of initial configuration such that the pseudotrajectory has a connected collision graph and a pathological collision.

As we suppose that the collision graph is connected, we can construct a clustering tree $T := (q_i, \bar{q}_i)_{i\leq r-1}$ as in the proof of Proposition 5.2. We define τ_{path} the time of the first pathological collision. The corresponding collision can either create a loop in the collision graph or be a clustering multiple interaction.

The first case can be treaten as a recollision, which is already treaten in the proof of Proposition 5.2 and in the preceding Section, we have

$$\int_{\mathbb{T}^{r-1}\times B_r(\mathbb{V})}\mathbb{1}_{\mathscr{R}^{\mathrm{reco}}_{(\lambda_1,\cdots\lambda_{\mathfrak{l}})}}\frac{e^{-\mathscr{H}_r}}{(2\pi)^{\frac{rd}{2}}}\,\mathrm{d}X_{2,r}\,\mathrm{d}V_r \leq \frac{C_r\delta^{r-2}}{(\mu\mathfrak{d})^{r-1}}\varepsilon^{1/4}.$$

In the second case, there are two clustering collisions $j < \tilde{j}$ such that $\{q_j, \bar{q}_j\} \cap \{q_{\tilde{j}}, \bar{q}_{\tilde{j}}\}$ and

$$\forall \tau \in (\tau_j, \tau_{\tilde{j}}),\ |\mathsf{x}_{q_j}(\tau) - \mathsf{x}_{\bar{q}_j}(\tau)| < \varepsilon.$$

Two particles (q_j, q_j') can touch on a time interval shorter than $\frac{\varepsilon}{|v_{q_j}(\tau_j)-v_{\bar{q}_j}(\tau_j)|}$ (which is integrable). Hence, using the same strategy than in the proof of Proposition 5.2:

$$\int_{\mathbb{T}^{r-1}\times B_r(\mathbb{V})} \mathbb{1}_{\mathscr{R}^{\mathrm{mult}}_{(\lambda_1,\cdots\lambda_\mathfrak{l})}} \frac{e^{-\mathscr{H}_r}}{(2\pi)^{\frac{rd}{2}}}\, \mathrm{d}X_{2,r}\, \mathrm{d}V_r \leq \frac{C_r\delta^{r-1}}{(\mu\mathfrak{d})^{r-1}}\varepsilon^{\mathfrak{a}}.$$

Summing on all the possible $(\lambda_1,\cdots\lambda_\mathfrak{l})$ we obtain the expected result. □

Proof of (8.10). We have now to prove the Estimation (8.10):

$$\int_{\mathbb{T}^{r-1}\times(\mathbb{R}^d)^r} \mathbb{1}_{\mathscr{O}_\varpi} \mathbb{1}_{\substack{Z_r \text{ form a} \\ \text{possible cluster}}} \frac{e^{-\frac{1}{2}\mathscr{H}_r(Z_r)}}{(2\pi)^{dr/2}}\, \mathrm{d}X_{2,r}\, \mathrm{d}V_r \leq \frac{C_r\delta^{\min\{2,r-1\}}}{(\mu\mathfrak{d})^{r-1}}\varepsilon^{\mathfrak{a}},$$

Without loss of generality, we suppose that $1 \in \varpi$.

Fix the family $\omega_1,\cdots,\omega_p$ of subset covering $[n]$, and $(\underline{\lambda}_i)_{i\leq p} = (\lambda_i^1,\cdots,\lambda_i^{l_i})_{i\leq p}$ a family of partition of ω_i such that the union of the collision graph associated to parameters $((\lambda_i^j)_j)_i$ is connected.

We begin by fix Z_ϖ and sum the clustering of the particle in $[n]\setminus\varpi$

$$\int_{\mathbb{T}^{n-1}\times(\mathbb{R}^d)^n} \mathbb{1}_{\mathscr{O}_\varpi} \mathbb{1}_{\substack{Z_r \text{ form a} \\ \text{possible cluster}}} \frac{e^{-\frac{1}{2}\mathscr{H}_n(Z_n)}}{(2\pi)^{dn/2}}\, \mathrm{d}X_\varpi\, \mathrm{d}V_\varpi \leq \mathbb{1}_{\mathscr{O}_\varpi} \frac{e^{-\frac{1}{2}\mathscr{H}_{|\varpi|}(Z_\varpi)}}{(2\pi)^{d|\varpi|/2}} \frac{C_n\delta^{n-|\varpi|}}{(\mu\mathfrak{d})^{|\varpi|}}\varepsilon^{\mathfrak{a}}.$$

Then we integrate with respect to $\mathrm{d}X_{\varpi\setminus\{1\}}\,\mathrm{d}V_\varpi$. □

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