LONG TIME VALIDITY OF THE LINEARIZED BOLTZMANN EQUATION FOR HARD SPHERES: A PROOF WITHOUT BILLIARD THEORY

CORENTIN LE BIHAN

ABSTRACT. We study space-time fluctuations of a hard sphere system at thermal equilibrium, and prove that the covariance converges to the solution of a linearized Boltzmann equation in the low density limit, globally in time. This result has been obtained previously in [7], by using uniform bounds on the number of recollisions of dispersing systems of hard spheres (as provided for instance in [9]). We present a self-contained proof with substantial differences, which does not use this geometric result. This can be regarded as the first step of a program aiming to derive the fluctuation theory of the rarefied gas, for interaction potentials different from hard spheres.

CONTENTS

UMPA (UMR CNRS 5669), ÉCOLE NORMALE SUPERIEUR DE LYON, 46 ALLÉE D'ITALIE, 69364 LYON, FRANCE

E-mail address: corentin.le-bihan@ens-lyon.fr.

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1. Introduction

Consider a system of N classical particles in a box $\Lambda \subset \mathbb{R}^d$ ($d \geq 3$), interacting by means of a two-body potential $\mathcal{V}_\varepsilon(\cdot) := \mathcal{V}(\cdot/\varepsilon)$. We are interested in the behavior of the system as the number of particles goes to infinity and the interaction length scale ε is fixed by the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$. It is a limit of low density where the mean free path of a particle between two collisions is of order $O(1)$.

Away from equilibrium, it is expected that the system is governed by the Boltzmann equation in the low density limit. However most of the existing rigorous results are valid for short time, such that only a small fraction of the particles actually interact. The first convergence proofs were provided in the fundamental work of Lanford [18] for hard spheres and by King [17] for different finite range potentials (see also [11, 26]). Quantitative convergence bounds have been obtained later on (see [14, 20]).

Illner and Pulvirenti proved a first long time convergence result in [15] (see also [12]), but only for a very diluted gas in the whole space, where dispersion is the dominant phenomena. Other long time results have been obtained later on for a system of one labeled particle evolving in a background at equilibrium (see [27] for arbitrary kinetic times and [3] for diffusive times). The law of the tagged particle follows then the linear Boltzmann equation. See also [2, 10] for adaptations of the proof to interaction potentials different from hard core.

Looking at a tagged particle in a background at equilibrium can be seen as a perturbation of order $O(1)$ of the equilibrium measure. The next natural step is to study small fluctuations around equilibrium which can be seen as perturbations of order $O(N)$ (we are interested in the square of the small fluctuations). Note that a reasonable "final step" would be to understand on long time non equilibrium chaotic measures which are $O(C^N)$ perturbations.

In the low density limit, the fluctuations behave like a Gaussian field with covariance governed by the linearized Boltzmann equation, as predicted in [24, 25]. The rigorous proof is separated in two main parts: first the convergence of the covariance and second checking asymptotically the Wick's rules characterizing the higher order moments; treated first for short times, respectively in [24], and [5, 6] in the more general context of non equilibrium states. Concerning the global in time result, the Wick's rule has been treated recently in the case of hard spheres in [8]. Convergence of the covariance has been obtained first for hard disks in dimension 2 in the canonical ensemble (see [4]), using that the partition function is uniformly bounded (in ε), which is a specificity of dimension 2. Later on a proof has been given for dimension 3 in [7], in the grand canonical ensemble.

The purpose of the present work is to propose a different method of proof for the result in [7]. As known, a crucial part of the argument leading to the Boltzmann equation amounts to showing that dynamical memory effects (called recollisions) are vanishing in the limit. The long time result is based then on a sampling checking the trajectories carefully and eliminating the recollisions on very small time scales (of order δ, a power of ε). On these scales, it is used in [7] that the dynamics is decomposed on independent clusters of finite size, each of which behaves as a dispersing billiard with uniformly bounded number of collisions. The latter property is unproved (possibly false) for arbitrary potentials with compact support (defining, say, a collision as a two-by-two interaction at distance ε). Even in the case of hard spheres, the property is delicate: explicit bounds have been provided in [9] by means of refined geometric techniques.

This motivates us to develop a different argument circumventing any uniform control on recollision numbers. The main ingredients are a subtle conditioning of the initial data forbidding explosions of the number of recollisions, together with a suitable dynamical cumulant decomposition method, inspired by [6].

1.1. **Definition of the system.** Let $\Lambda := \mathbb{R}^d/\mathbb{Z}^d$ (with $d \geq 3$) be the domain. We denote $\mathbb{D} = \Lambda \times \mathbb{R}^d$ its tangent bundle and $\mathcal{D}^n_\varepsilon \subset \mathbb{D}^n$ the n-particle canonical phase space:

$$\mathcal{D}^n_\varepsilon := \Big\{ Z_n := (x_1, v_1, \cdots, x_n, v_n) \in \mathbb{D}^n,\ \text{for } 1 \leq i < j \leq n,\ |x_i - x_j| > \varepsilon \Big\}. \tag{1.1}$$

Here and in the following, we use the notation

$$X_n = (x_1, \cdots, x_n),\ V_n = (v_1, \cdots, v_n),\ \text{and } z_i = (x_i, v_i).$$

On each $\mathcal{D}^n_\varepsilon$ we construct the hard sphere dynamics as the Hamiltonian dynamics associated with the Hamiltonian

$$\mathcal{H}^\varepsilon_n(Z_n) := \frac{1}{2}\|V_n\|^2 + \mathcal{V}^\varepsilon_n(X_n),\quad \mathcal{V}^\varepsilon_n(X_n) := \sum_{1 \leq i < j \leq n} \mathcal{V}\left(\frac{\|x_i - x_j\|}{\varepsilon}\right) \tag{1.2}$$

where $\mathcal{V}$ is the hard core interaction potential

$$\mathcal{V}(r) := \begin{cases} 0 \text{ if } |r| > 1 \\ \infty \text{ else} \end{cases}. \tag{1.3}$$

In this dynamics particles move along straight lines until they meet each other. If at time τ we have $|x_q(\tau) - x_{q'}(\tau)| = \varepsilon$, the outgoing velocities are given by the following scattering law:

$$\begin{cases} v_q(\tau^+) &= v_q(\tau^-) - \dfrac{x_{q'}(\tau) - x_q(\tau)}{|x_{q'}(\tau) - x_q(\tau)|} \cdot \Big(v_q(\tau^-) - v_{q'}(\tau^-)\Big) \dfrac{x_{q'}(\tau) - x_q(\tau)}{|x_{q'}(\tau) - x_q(\tau)|} \\ v_{q'}(\tau^+) &= v_{q'}(\tau^-) + \dfrac{x_{q'}(\tau) - x_q(\tau)}{|x_{q'}(\tau) - x_q(\tau)|} \cdot \Big(v_q(\tau^-) - v_{q'}(\tau^-)\Big) \dfrac{x_{q'}(\tau) - x_q(\tau)}{|x_{q'}(\tau) - x_q(\tau)|}. \end{cases} \tag{1.4}$$

This process is well defined for all times, almost everywhere in $\mathcal{D}^n_\varepsilon$ with respect to the Lebesgue measure (see [1]).

We denote in the following $\mathcal{D}_\varepsilon := \bigsqcup_{n \geq 0} \mathcal{D}^n_\varepsilon$ the grand canonical phase space and $\mathcal{N}$ the random number of particles.We can then extend the Hamiltonian dynamics to $\mathcal{D}_\varepsilon$ and denote $\mathbf{Z}_\mathcal{N}(t)$ the realization (defined almost surely) of the hard sphere flow on $\mathcal{D}_\varepsilon$ with random initial data $\mathbf{Z}_\mathcal{N}(0)$: for $\mathcal{N} = n$, $\mathbf{Z}_\mathcal{N}(t)$ follows the Hamiltonian dynamics on $\mathcal{D}^n_\varepsilon$.

The initial data is sampled according to the stationary measure introduced next. The *grand canonical Gibbs measure* $\mathbb{P}_\varepsilon$ (and its expectation $\mathbb{E}_\varepsilon$) are defined on $\mathcal{D}_\varepsilon$ as follows. An application $G : \mathcal{D}_\varepsilon \to \mathbb{R}$ is a test function if there exists a sequence $(g_n)_{n \geq 0}$ with $g_n \in L^\infty(\mathbb{D}^n)$ and

$$\text{for } \mathcal{N} = n,\ G(\mathbf{Z}_\mathcal{N}) := g_n(\mathbf{Z}_\mathcal{N}).$$

Then we define $\mathbb{E}_\varepsilon$ as

$$\mathbb{E}_\varepsilon[G(\mathbf{Z}_\mathcal{N})] := \frac{1}{\mathcal{Z}_\varepsilon} \sum_{n \geq 0} \frac{\mu^n_\varepsilon}{n!} \int_{\mathbb{D}^n} g_n(Z_n)\, \frac{e^{-\mathcal{H}^\varepsilon_n(Z_n)}}{(2\pi)^{nd/2}} dZ_n, \tag{1.5}$$

where $\mathcal{Z}_\varepsilon$ is a normalisation constant and μ_ε is tuned to respect the Boltzmann-Grad scaling $\mu_\varepsilon \varepsilon^{d-1} = 1$.

The empirical distribution at time t is defined as the average configuration of particles at time t: for g some test function on $\mathbb{D}$,

$$\pi^\varepsilon_t(g) := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} g(\mathbf{z}_i(t)). \tag{1.6}$$

At equilibrium, we have the following law of large numbers. Denote

$$M(v) := (2\pi)^{-d/2} e^{-\|v\|^2/2}. \tag{1.7}$$

Theorem 1.1. *For any continuous and bounded test function $g : \Lambda \times \mathbb{R}^d \to \mathbb{R}$, for all $t \in \mathbb{R}$ and for any $a > 0$,*

$$\lim_{\varepsilon \to 0} \mathbb{P}_\varepsilon \left[\left| \pi_\varepsilon^t(g) - \int g(z) M(v) dz \right| \geq a \right] = 0. \tag{1.8}$$

Remark 1.1. *The previous result is a simple corollary of the Lanford's theorem and of the invariance of the measure (see* [18]*).*

1.2. **Convergence to the linearized Boltzmann equation.** The aim of this article is to investigate the next order, namely the fluctuation field

$$\zeta_\varepsilon^t(g) := \mu_\varepsilon^{1/2} \Big(\frac{1}{\mu_\varepsilon} \sum_{1 \leq i \leq \mathcal{N}} g(\mathbf{z}_i(t)) - \mathbb{E}_\varepsilon[\pi_0^\varepsilon(g)] \Big). \tag{1.9}$$

When ε tends to 0, collisions become rare and we expect that particles can see each other only a finite number of times in any bounded time interval. We define the linearized Boltzmann operator as

$$\mathcal{L} g(v) := \int_{\mathbb{S}^{d-1} \times \mathbb{R}^d} \big(g(v') + g(v'_*) - g(v) - g(v_*) \big) ((v - v_*) \cdot \eta)_+ M(v_*) d\eta \, dv_*, \tag{1.10}$$

where (v', v'_*) are given by the scattering of (v, v_*, η)

$$\begin{cases} v' & := v - \eta \cdot (v - v_*) \eta \\ v'_* & := v_* + \eta \cdot (v - v_*) \eta. \end{cases} \tag{1.11}$$

This operator describes the variation of mass due to changes of velocity of colliding particles. The operator $\mathcal{L}$ is a self-adjoint negative operator on $L^2(M(v)dz)$. We want to prove the following result

Theorem 1.2. *Let $f, g \in L^2(M(v)dz)$ be two test functions. Then we have the following convergence result: for all $t \geq 0$,*

$$\mathbb{E}_\varepsilon \left[\zeta_\varepsilon^t(h) \zeta_\varepsilon^0(g) \right] \underset{\varepsilon \to 0}{\longrightarrow} \left\langle h, e^{t(-v \cdot \nabla_x + \mathcal{L})} g \right\rangle$$

where $<,>$ is the Hermitian product on $L^2(M(v)dz)$.

Since the two bilinear operators

$$(h, g) \mapsto \mathbb{E}_\varepsilon \left[\zeta_\varepsilon^t(h) \zeta_\varepsilon^0(g) \right], \ (h, g) \mapsto \left\langle h, e^{t(-v \cdot \nabla_x + \mathcal{L})} g \right\rangle$$

are both continuous on $L^2(M(v)dz)$ (see [7]), it sufficient to prove Theorem 1.2 in a dense subset. This also allows to have a quantitative version of the theorem, which we state for completeness.

We define for g smooth the norm

$$\|g\| := \sup_{(x,v) \in \mathbb{D}} \left| M^{-1}(v) g(x, v) \right| \tag{1.12}$$

and we consider test functions g such that

$$\|g\| + \| \, |\nabla_x g| \, \| < \infty. \tag{1.13}$$

Theorem 1.3. *Let g and h two $C^1(\mathbb{D})$ functions satisfying condition (1.13). Then there exist three constants $C>1$, $C'>1$ and $\alpha\in(0,1)$ independent of g,h such that for any ε small enough, $T>1$, $\theta<\frac{1}{C'T^2}$*

$$\begin{aligned}\sup_{t\in[0,T]}&\left|\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right]-\left\langle h,e^{t(-v\cdot\nabla_x+\mathcal{L})}g\right\rangle\right|\\&\leq C\Big(CT^{3/2}\theta^{1/2}+(CT)^{2^{T/\theta}}\varepsilon^\alpha\Big)\|h\|\big(\|g\|+\|\nabla g\|\big).\end{aligned}\tag{1.14}$$

In particular we can choose $T=o((\log|\log\varepsilon|)^{1/3})$ and $\theta=\frac{1}{\beta\log|\log\varepsilon|}$, $\beta\in(0,1)$ small enough.

Notations. From now on we will use the following notations.

We denote for $m<n$ two integers, $[m,n]:=\{m,m+1,\cdots,n\}$ and $[n]:=[1,n]$.

For $Z_n\in\mathbb{D}^n$, and $\omega\subset[n]$, we denote

$$Z_\omega:=(z_{\omega(1)},\cdots,z_{\omega(|\omega|)})$$

where $\omega(i)$ is the i-th element of ω counted in increasing order. For $1\leq l<m\leq n$, $Z_{l,m}:=Z_{[l,m]}$.

Given a family of particles indices $\{i_1,\cdots,i_n\}$, the notation $(i_1,\cdots,i_n)$ indicates the ordered sequence in which $\forall k\neq l$, $i_k\neq i_l$. In addition

- $\underline{i}_n:=(i_1,\cdots,i_n)$,
- for $m\leq n$, $\underline{i}_m=(i_1,\cdots,i_m)$, and more generally for $\omega\subset[1,n]$, $\underline{i}_\omega:=(i_{\min\omega},\cdots,i_{\max\omega})$,
- for $0\leq m<n$ and $(i_1,\cdots,i_m)$, $\sum\limits_{(i_{m+1},\cdots,i_n)}$ denotes the sum over every family such that for $k<l\leq n$, $i_k\neq i_l$,
- $\mathbf{Z}_{\underline{i}_n}:=(\mathbf{z}_{i_1},\cdots,\mathbf{z}_{i_n})$, as ordered sequence.

We also precise the sense of Landau notation: $A=B+O(D)$ means that there exists a constant C depending only on the dimension such that $|A-B|<C\,D$.

Finally let h_n be a function on $\mathbb{D}^n$. We denote

$$\mathbb{E}_\varepsilon\big[h_n\big]:=\mathbb{E}_\varepsilon\left[\frac{1}{\mu_\varepsilon^n}\sum_{(i_1,\cdots,i_n)}h_n\big(\mathbf{Z}_{\underline{i}}\big)\right]$$

and the associated truncated function defined on $\mathcal{D}_\varepsilon$

$$\hat{h}_n(\mathbf{Z}_\mathcal{N}):=\frac{1}{\mu_\varepsilon^n}\sum_{(i_1,\cdots,i_n)}h_n\big(\mathbf{Z}_{\underline{i}_n}\big)-\mathbb{E}_\varepsilon\big[h_n\big].$$

1.3. **Strategy of the proof.** We explain now the main ideas of the proof and of the improvement with respect to [7].

Because $\zeta^t_\varepsilon(g)$ is a centered random variable,

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right]=\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{i=1}^{\mathcal{N}}h(\mathbf{z}_i(t))\,\zeta^0_\varepsilon(g)\right].\tag{1.15}$$

The first step is to find a family of functionals $\Phi^t_{\varepsilon,n}:L^\infty(\mathbb{D})\to L^\infty(\mathbb{D}^n)$ corresponding to the pullback of the test function h at time 0

$$\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{i=1}^{\mathcal{N}}h(\mathbf{z}_i(t))\,\zeta^0_\varepsilon(g)\right]=\sum_{n\geq1}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_n)}\Phi^t_{\varepsilon,n}[h]\big(\mathbf{Z}_{\underline{i}_n}(0)\big)\,\zeta^0_\varepsilon(g)\right].\tag{1.16}$$

It turns out that the $\Phi^t_{\varepsilon,n}[h]$ are a sum over *histories*. Loosely speaking, a history is defined as a way to remove (or not) particles at each collision, so that at time t there remains only one particle (see the picture below). Then

$$\Phi^t_{\varepsilon,n}[h](Z_n) := \frac{1}{n!}\sum_{\text{history}} h(z_k(t))\mathbb{1}_{\text{history}}\sigma_{\text{history}} \tag{1.17}$$

where $\sigma_{\text{history}} = \pm 1$ and $z_k(t)$ is the position of the last particle k (k depends of the history). This formula will be explained precisely in section 2. For the moment, we mention that the signs σ_{history} are related to a splitting of collision operators into a positive and negative part (as in (1.10)).

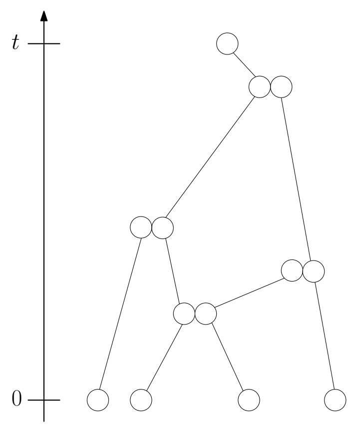

FIGURE 1. Example of history for four particles.

The classical method to prove convergence of a hard sphere system to the Boltzmann equation (and here to the linearized equation) amounts to show that each term of the sum (1.16) converges to its formal limit. This is the way we compare the hard sphere process with the limit punctual process. In this procedure, it is natural to separate a principal part containing a controlled number of collisions, from some rest terms encoding ill-behaved trajectories (for instance trajectories with more than $n-1$ collisions, which do not have a counterpart in the limit process).

For the argument to be rigorous, we then need a bound on the rest terms of the sum. In usual derivations of the Boltzmann equation (see for instance [18, 17, 14, 3, 20]) one resorts to L^∞ bounds and to a dual representation of the sum (1.16). In contrast here we rely on the above pullback formula, together with suitable stopping times t_{s} truncating the formula when the number of histories becomes uncontrolled. To implement this idea it is convenient to consider L^2 bounds as in [4, 7]. Indeed (using notation introduced at the end of the previous section), because $\zeta^0_\varepsilon(g)$ is centered

$$\begin{aligned}\left|\mathbb{E}_\varepsilon\left[\sum_{\underline{i}_n}\Phi^{t-t_s}_{\varepsilon,n}[h](\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\right]\right| &\leq \mathbb{E}_\varepsilon\left[\left(\mu^n_\varepsilon\widehat{\Phi^{t-t_{\text{s}}}_{\varepsilon,n}}[h](\mathbf{Z}_{\underline{i}_n}(t_{\text{s}}))\right)^2\right]^{\frac{1}{2}}\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac{1}{2}}\\ &\equiv \mathbb{E}_\varepsilon\left[\left(\mu^n_\varepsilon\widehat{\Phi^{t-t_{\text{s}}}_{\varepsilon,n}}[h](\mathbf{Z}_{\underline{i}_n}(0))\right)^2\right]^{\frac{1}{2}}\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac{1}{2}}\end{aligned}$$

using Cauchy-Schwartz and the invariance of the Gibbs measure. By virtue of such estimates, we do not need to take into account what happens for pathological histories before t_s.

Unfortunately, in the bound for $\Phi^t_{\varepsilon,n}[h]$, we do not know how to take into account the cancellations due to the signs in σ_{history}. Thus we have to count the number of possible histories and collisions. We then need to distinguish two kinds of collisions: those where one particle is removed, and those where both particles are kept, called *recollisions*. The second type is harder to control.

We need two different samplings to control each type of collision separately. The sampling method (already used in [7]) is an adaptation of [3, 4] (and reminiscent of [13] in the context of the quantum Lorentz gas).

The first sampling has a relative large step $\theta = 1/\beta \log|\log\varepsilon|$ (with $\beta \in (0,1)$ set later) and enables to control a moderate growth of collisions with removal. This will be the source of the slow speed of convergence in (1.14).

The second sampling, which has a shorter step $\delta = \varepsilon^{\beta'}$ (with $\beta' \in (0,1)$ set later) is used to control possibly many recollisions on the short time scale. These collisions will be allowed only on the last time interval $[t_s, t_s+\delta]$.

In the present paper, two conditionings on initial data are used, to deal with the recollision problem. The first one is symmetric on all the particles and forbids a group of more than a fixed integer $\gamma > 0$ to interact altogether on each small time interval $[k\delta, (k+1)\delta]$ (for $k \in \mathbb{N}$). Once applied this conditioning on the invariant measure, the paper [7] uses the billiard theory developed in [9] to control the number of histories in clusters of γ particles. Notice that such result has no known analogue for different interaction potentials, even with compact support.

The main goal of this paper is to avoid the latter geometrical argument, as explained next.

Let us define the collision graph of a trajectory on a time interval $[\tau, \tau']$ as the graph where the vertices are the set of particles and to each collision happening on $[\tau, \tau']$ corresponds an edge between the colliding particles. A trajectory on the time interval $[t_s, t]$ is said *non-pathological* (see the figure) if

- its collision graph restricted to $[t_s+\delta, t]$ is a tree (at each collision, one particle is removed),
- on $[t_s, t_s+\delta]$ the collision graph has no cycle (but there can be recollisions).

Due to the symmetric conditioning, one particle can meet at most γ other particles on $[t_s, t_s+\delta]$, and thus there are less than γ recollisions per particle. Therefore the number of non-pathological trajectories and corresponding recollisions is controlled by construction.

We then introduce a second conditioning forbidding pathological trajectories. One difficulty is that this conditioning will introduce asymmetry. Since there are approximately μ_ε particles in the system, choosing one particle costs roughly μ_ε and in a symmetric conditioning the choice of k particles would cost μ_ε^k. But the sum $\sum_{\underline{i}_n} \Phi_{\varepsilon,n}[h](\mathbf{Z}_{\underline{i}_n})$ is already a sum over n chosen particles, and for each term of this sum we are interested in particles of the "background" which can influence the n selected particles. Hence, it is sufficient to impose an asymmetric conditioning where in the set of k particles producing a pathology, the first one is chosen first in $\underline{i}_n$, and then the $k-1$ other particles are chosen in the whole set of particles. Such procedure will provide a gain of μ_ε^{-1} which turns out to be enough to control the error term, by means of a cumulant expansion.

We conclude by describing better the asymmetric conditioning, which is the main novel tool of this paper. Let $\chi(Z_r)$ be the indicator function which takes value 1 if there exist history parameters such that the graph on $[t_s, t_s+\delta]$ with initial data Z_r at time t_s has a cycle and is connected. Because the indicator function involves a bounded number of particles, its weight $\|\chi(Z_r)\|_{L^1}$ is small. We then introduce an asymmetric conditioning $\mathcal{X}_{\underline{i}_n}(\mathbf{Z}_{\mathcal{N}}(t_s))$ imposing the existence of a set

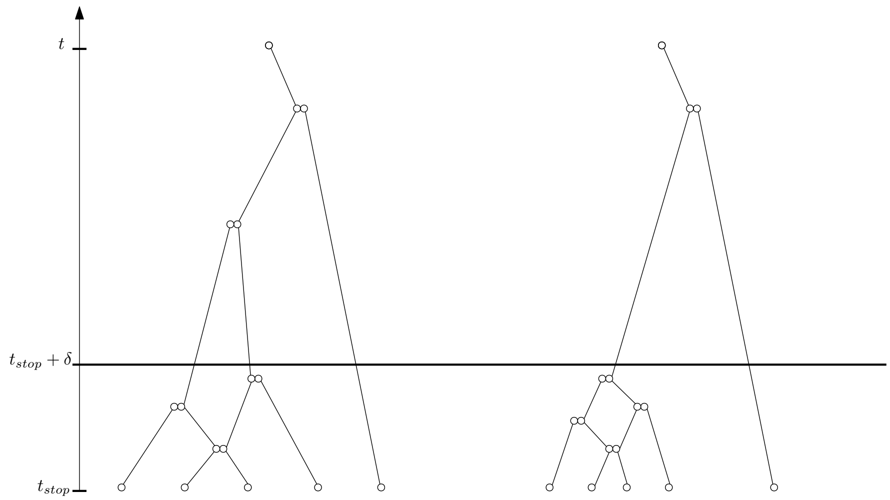

FIGURE 2. An example of one non-pathological trajectory (on the left) and a pathological one (on the right).

of particles ω containing at least one particle of $\{i_1, \cdots, i_n\}$ such that $\chi(\mathbf{Z}_\omega(t_s)$ is equal to 1, *id est* one trajectory containing a particle of $\underline{i}_n$ is pathological.

Let us give an idea of how to bound $\mathcal{X}_{\underline{i}_n}(\mathbf{Z}_\mathcal{N}(t_s))$. We develop the constraint over finite numbers of background variables

$$\mathcal{X}_{\underline{i}_n}(\mathbf{Z}_\mathcal{N}(t_s)) = \sum_{p\geq n} \sum_{(i_{n+1},\cdots,i_p)} \mathfrak{X}_{n,p}\big(\mathbf{Z}_{\underline{i}_p}(t_s)\big).$$

The $\mathfrak{X}_{n,p}\big(\mathbf{Z}_{\underline{i}_p}(t_s)\big)$ can be expressed as sums over families of particles $(\omega_1, \cdots, \omega_k)$, where ω_i is a subset of $\underline{i}_p$, of terms

$$(-\chi(\mathbf{Z}_{\omega_1}(t_s)))(-\chi(\mathbf{Z}_{\omega_2}(t_s)))\cdots(-\chi(\mathbf{Z}_{\omega_k}(t_s))).$$

The ω_i can intersect, hence the number of terms in $\mathfrak{X}_{n,p}$ is huge. But the (first) symmetric conditioning permits to bound the number of intersecting sets. If $\omega_1, \cdots, \omega_k$ intersect, all the particles in their union are close. Hence the size of $\omega_1 \cup \cdots \cup \omega_k$ is bounded by γ and k is smaller than 2^γ. This is sufficient to bound $\mathfrak{X}_{n,p}$.

The paper is organized as follows. In section 2 we give a proper definition of history and we use it to construct the functionals $\Phi^t_{\varepsilon,n}$. Then the two samplings mentioned above are implemented, and the conditionings applied. This allows to decompose $\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right]$ into a main term, plus error terms of different nature: a development on trajectories (actually called below pseudotrajectories) (i) without recollisions (bounded in Section 4), (ii) with non-pathological recollisions (bounded in section 5) and (iii) with pathological recollisions (bounded in Section 6). The estimation of the error terms requires standard $L^2(\mathbb{P}_\varepsilon)$ estimates based on static cumulant decomposition, which are reported in Section 3. Finally, the convergence of the main term is proved in Section 7.

2. Development along pseudotrajectories and time sampling

2.1. **Definition of (forward) pseudotrajectories.** Consider n particles. To lighten notation for pseudotrajectories (called "histories" in the introduction), we will drop their dependence on ε.

For $m \leq n$, fix a family of *pseudotrajectory parameters*

$$((s_i, \bar{s}_i)_{1\leq i\leq n-m}, (\kappa_j)_{1\leq j\leq n}) \in \{(\pm 1, \pm 1)\}^{n-m} \times \mathbb{N}^n,$$

and an initial data $Z_n \in \mathcal{D}_\varepsilon^n$.

We construct iteratively the pseudotrajectories $\mathrm{Z}_n(\tau, ((s_i, \bar{s}_i)_{1\leq i\leq n-m}, (\kappa_j)_{1\leq j\leq n}), Z_n)$, the collision indices $\iota(\tau)$ and recollision indices $(\kappa_j(\tau))_{1\leq j\leq n}$. At time $\tau = 0$, we set $\iota(0) := 1$ and for all j, $\kappa_j(0) := \kappa_j$. Moreover at $\tau = 0$, $\mathrm{Z}_n(0) = Z_n \in \mathcal{D}_\varepsilon^n$. The number of particles decreases with time and is equal to $(n + 1 - \iota(\tau))$.

If $\iota(\tau) < n - m + 1$, the remaining particles move freely along straight lines, until there is a new collision between two of them at time τ, say q and q' with $q < q'$. If $s_{\iota(\tau^-)} = 1$ (respectively -1), we look at $\kappa_q(\tau^-)$ (respectively $\kappa_{q'}(\tau^-)$):

- if it is strictly positive, we have a *recollision*. The two particles scatter as in (1.4) and $\kappa_q(\tau^+) = \kappa_q(\tau^-) - 1$ (respectively $\kappa_{q'}(\tau^+) = \kappa_{q'}(\tau^-) - 1$),
- if it is 0 we have an *annihilation*: we remove the particle q (in the case where $s_{\iota(\tau^-)} = -1$, we remove q'). The other particle q' (respectively q) scatters if $\bar{s}_{\iota(\tau^-)} = 1$ or continues freely along straight line else.

Finally we increment $\iota(\tau)$.

When $\iota(\tau) = n - m + 1$ (there are m particles left), all the annihilations have been performed and the remaining particles evolve according to the Hamiltonian flow.

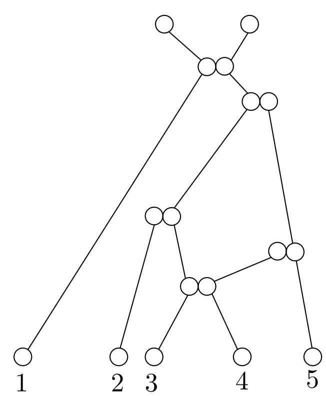

FIGURE 3. Pseudotrajectory associated with $(((1, -1), (1, 1), (-1, 1)), (0, 0, 1, 0, 0))$.

Let ω be a finite subset of $\mathbb{N}^*$. We will denote $\mathrm{Z}_\omega(t, Z_\omega, ((s_i, \bar{s}_i)_{i\leq|\omega|-m}, (\kappa_j)_{j\in\omega}))$ the pseudotrajectory with particles of ω and $\mathrm{Z}_\omega(t)$ when there is no ambiguity on the parameters. Note that this should not be confused with $\mathbf{Z}_\omega(t)$, the configuration of the particles ω in the realization of the hard sphere flow over $\mathcal{D}_\varepsilon$ (the "real trajectories").

Definition 2.1 (Collision graph). Given $Z_r \in \mathcal{D}_\varepsilon^r$ and parameters $((s_i, \bar{s}_i), (\kappa_j)_j)$, we construct the *collision graph* $\mathcal{G}_r^{[0,t]}$ as the couple (E, V), with $V := \{1, \cdots, r\}$ and

$$E \subset \{(i, j)_\tau, \text{ where } (i, j) \in V^2, \, i < j, \, \tau \in [0, t]\}$$

such that $(i, j)_\tau \in E$ if and only if there is a collision at time τ in the pseudotrajectory between particle i and j. It is an unoriented graph where the edges are labeled by the collision times.

By well known properties of the hard sphere dynamics (see [1]), for almost all Z_r, $\mathcal{G}_r^{[0,t]}$ has a finite number of edges. We can order $\tau_1 < \tau_2 < \cdots < \tau_k$ the collision times of $\mathcal{G}_r^{[0,t]}$.

In the following we possibly denote $E\left(\mathcal{G}_r^{[0,t]}\right) := E$.

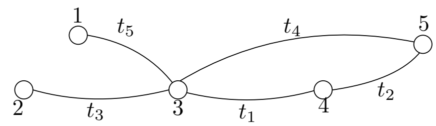

FIGURE 4. Collision graph associated with the pseudotrajectory of Figure 3, with $t_1 < t_2 < t_3 < t_4 < t_5$ the collision times.

2.2. **Development along pseudotrajectories.** The pseudotrajectories are used to pull back a function evaluated at time t, up to a previous time 0.

Let $m \leq n$ be two integers, and $((s_i, \bar{s}_i)_{1\leq i\leq n-m}), (\kappa_j)_{1\leq j\leq n})$ be collision parameters and $t > 0$ the finite time. In order not to count twice the same pseudotrajectory, all parameters have to be taken into account. We define $\mathcal{R}^{m\leftarrow n,t}_{((s_i,\bar{s}_i),(\kappa_j))} \subset \mathcal{D}^n_\varepsilon$ as the set of initial parameters Z_n such that at time t, the following condition is verified: $\{1, \cdots, m\}$ are the remaining particles of $\mathrm{Z}_n(t, Z_n)$, and the recollision indices defined in the previous section vanish: for all j, $\kappa_j(t) = 0$.

Let $h_m \in L^\infty(\mathbb{D}^m)$ be a test function (not necessarily symmetric under permutation of variables). We define the *pseudotrajectory development* as the functional $\Phi^t_{m\leftarrow n} : L^\infty(\mathbb{D}^m) \to L^\infty(\mathbb{D}^n)$ with

$$\Phi^t_{m\leftarrow n}[h_m](Z_n) := \frac{1}{(n-m)!}\sum_{\substack{(s_i,\bar{s}_i)_{i\leq n-m}\\ (\kappa_j)_j}} \left(\prod_{i=1}^{n-m} \bar{s}_i\right) \mathbb{1}_{\mathcal{R}^{m\leftarrow n,t}_{((s_i,\bar{s}_i),(\kappa_j))}} h_m(\mathrm{Z}_n(t, Z_n)). \tag{2.1}$$

We have the following semigroup property:

Proposition 2.1. *Consider $m \leq n$ two integers, $t > t' > 0$ two evaluation times and $\underline{i}_m$ a family of particles. Then for any function $h_m \in L^\infty(\mathbb{D}^m)$ and almost all initial data $Z_n \in \mathcal{D}^n_\varepsilon$,*

$$\sum_{(i_{m+1},\cdots i_n)} \Phi^t_{m\leftarrow n}[h_m](Z_n) = \sum_{n'=m}^{n}\sum_{(i_{m+1},\cdots,i_n)} \Phi^{t'}_{n'\leftarrow n}\left[\Phi^{t-t'}_{m\leftarrow n'}[h_m]\right](Z_n). \tag{2.2}$$

Proof. Fix collision parameters $((s_i, \bar{s}_i)_{i\leq n-m}, (\kappa_j)_{j\leq n})$ and an initial data. In the pseudotrajectory $\mathrm{Z}_n(\tau, ((s_i, \bar{s}_i), (\kappa_j)), Z_n)$, let ω be the set of remaining particles at time t' and t_l the time of the last annihilation before t'. We construct two sets of collision parameters

$$((s'_i, \bar{s}'_i)_{i\leq n-|\omega|}, (\kappa'_j)_{j\leq n}) := ((s_i, \bar{s}_i)_{i\leq n-|\omega|}, (\kappa_j - \kappa_j(t_l))_{j\leq n}),$$
$$((s''_i, \bar{s}''_i)_{i\leq |\omega|-m}, (\kappa''_j)_{j\in\omega}) := ((s_i, \bar{s}_i)_{i>|\omega|-m}, (\kappa_j - \kappa_j(t'))_{j\in\omega}).$$

We first prove the equality

$$\begin{aligned}\mathrm{Z}_n\left(t, ((s_i, \bar{s}_i)_{n-m}, (\kappa_j))_n, Z_n\right)& \\ = \mathrm{Z}_\omega&\left(t - t', ((s''_i, \bar{s}''_i)_{|\omega|-m}, (\kappa''_j)_\omega), \mathrm{Z}_n\left(t', ((s'_i, \bar{s}'_i)_{n-|\omega|}, (\kappa'_j)_n), Z_n\right)\right),\end{aligned}$$

with obvious simplification of notation. Until time t_l we have

$$\mathrm{Z}_n(\tau, ((s_i, \bar{s}_i)_n, (\kappa_j)_n), Z_n) = \mathrm{Z}_n(\tau, ((s'_i, \bar{s}'_i)_{n-|\omega|}, (\kappa'_j)_n), Z_n).$$

On the time interval $[t_l, t']$, there are only recollisions (no annihilation) in the two pseudotrajectories. On the left hand side, are treated by the decreasing parameters $\kappa_j(\tau)$ which decrease. On the right hand side, since we have treated all the annihilations, particles evolve along the Hamiltonian flow. Note that after time t_l, the $\kappa'_j(\tau)$ vanish. Hence at time t'

$$\mathrm{Z}_n(t', ((s_i, \bar{s}_i), (\kappa_j)), Z_n) = \mathrm{Z}_n(t', ((s'_i, \bar{s}'_i), (\kappa'_j)), Z_n)$$

and denoting $\iota(\tau)$ the collision indices associated with the first pseudotrajectory,

$$\iota(t') = n - |\omega|, \qquad \forall j,\ \kappa'_j = \kappa_j - \kappa_j(t').$$

Pursuing this construction we obtain that

$$\mathrm{Z}_n(t, ((s_i, \bar{s}_i), (\kappa_j)), Z_n) = \mathrm{Z}_\omega(t - t', ((s''_i, \bar{s}''_i), (\kappa''_j)), \mathrm{Z}_n(t', ((s'_i, \bar{s}'_i), (\kappa'_j)), Z_n))$$

which will be shortened as

$$\mathrm{Z}_n(t, Z_n) = \mathrm{Z}_\omega(t - t', \mathrm{Z}_n(t', Z_n)).$$

For each initial data, we have constructed an onto map

$$((s_i, \bar{s}_i)_n, (\kappa_j)_n) \mapsto (\omega, ((s'_i, \bar{s}'_i)_{n-|\omega|}, (\kappa'_j)_n), ((s''_i, \bar{s}''_i)_{|\omega|-m}, (\kappa''_j)_{|\omega|})$$

with in addition

$$\prod_{i=1}^{n-m} \bar{s}_i = \prod_{i=1}^{n-|\omega|} \bar{s}'_i \prod_{i=1}^{|\omega|-m} \bar{s}''_i.$$

Hence denoting $\mathcal{R}^{\omega\leftarrow n,t}_{((s'_i,\bar{s}'_i),(\kappa'_j))}$ the set of initial data such that the set of remaining particles of $\mathrm{Z}_n(t', ((s'_i, \bar{s}'_i), (\kappa'_j)), Z_n)$ is ω, and the corresponding recollision parameters $\kappa'_i(t')$ vanish, we have

$$\begin{aligned}
\sum_{\substack{(s_i,\bar{s}_i)_{i\leq n-m}\\(\kappa_j)_j}} \prod_{i=1}^{n-m} \bar{s}_i\, \mathbb{1}_{\mathcal{R}^{m\leftarrow n,t}_{((s_i,\bar{s}_i),(\kappa_j))}} h_m(\mathrm{Z}_n(t)) &= \sum_{[m]\subset\omega\subset[n]} \sum_{\substack{(s'_i,\bar{s}'_i)_{i\leq n-|\omega|}\\(\kappa'_j)_{j\leq n}}} \prod_{i=1}^{n-|\omega|} \bar{s}'_i\, \mathbb{1}_{\mathcal{R}^{\omega\leftarrow n,t}_{((s'_i,\bar{s}'_i),(\kappa'_j))}} \\
&\quad \times \sum_{\substack{(s''_i,\bar{s}''_i)_{i\leq|\omega|-m}\\(\kappa''_j)_\omega}} \prod_{i=1}^{|\omega|-m} \bar{s}''_i\, \mathbb{1}_{\mathcal{R}^{m\leftarrow\omega,t}_{((s''_i,\bar{s}''_i),(\kappa''_j))}} \big(\mathrm{Z}_n(t')\big) h_m\big(\mathrm{Z}_\omega\big((t-t', \mathrm{Z}_n(t'))\big)\big).
\end{aligned}$$

This proves that for $Z_n \in \mathcal{D}^n_\varepsilon$,

$$\begin{aligned}
&(n-m)!\Phi^t_{m\leftarrow n}[h_m](Z_n) \\
&\qquad = \sum_{n'=m}^{n} \sum_{\substack{\omega\subset[m+1,m+n]\\|\omega|=n'-m}} (n-n')!\Phi^{t'}_{n'\leftarrow n} \Big[(n'-m)!\Phi^{t-t'}_{m\leftarrow n'}[h_m]\Big] (Z_m, Z_\omega, Z_{\omega^c}).
\end{aligned}$$

Then summing on all families of particles

$$\sum_{(i_{m+1},\cdots i_n)} \Phi^t_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(0))$$
$$= \sum_{(i_{m+1},\cdots i_n)} \sum_{n'=m}^{n} \frac{1}{\binom{n-m}{n'-m}} \sum_{\substack{\omega\subset[m+1,m+n]\\|\omega|=n'-m}} \Phi^{t'}_{n'\leftarrow n}\left[\Phi^{t-t'}_{n\leftarrow n'}[h_m]\right]\big((\mathbf{Z}_{\underline{i}_m},\mathbf{Z}_{\underline{i}_\omega},\mathbf{Z}_{\underline{i}_{\omega^c}})(0)\big)$$
$$= \sum_{(i_{m+1},\cdots,i_n)} \sum_{n'=m}^{n} \Phi^{t'}_{n'\leftarrow n}\left[\Phi^{t-t'}_{n\leftarrow n'}[h_m]\right](\mathbf{Z}_{\underline{i}_n}(0)).$$

□

We can now write the pullback of a test function in terms of pseudotrajectory developments. This is the main result of this section:

Theorem 2.2. *Let $(i_1,\cdots,i_m)$ be a family of particles, with $i_{\max} := \max\{i_1,\cdots,i_m\}$. For almost all $\mathbf{Z}_\mathcal{N}\in\mathcal{D}_\varepsilon\cup\{\mathcal{N}\geq i_{\max}\}$ we have*

$$h_m\left(\mathbf{Z}_{(i_1,\cdots,i_m)}(t)\right) = \sum_{n\geq m}\sum_{(i_{m+1},\cdots,i_n)} \Phi^t_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)). \tag{2.3}$$

In addition if we do not fix $(i_1,\cdots,i_m)$ we have

$$\sum_{\underline{i}_m} h_m\left(\mathbf{Z}_{\underline{i}_m}(t)\right) = \sum_{n\geq m}\sum_{\underline{i}_n} \Phi^t_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)). \tag{2.4}$$

Proof. The proof is an adaptation of [21].

We rewrite the semigroup property of the previous proposition for a specific realization of the hard sphere process: for $t>t'>0$,

$$\sum_{(i_{m+1},\cdots i_n)} \Phi^t_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)) = \sum_{n'=m}^{n}\sum_{(i_{m+1},\cdots,i_{n'})}\sum_{(i_{n'+1},\cdots,i_n)} \Phi^{t'}_{n'\leftarrow n}\left[\Phi^{t-t'}_{n\leftarrow n'}[h_m]\right](\mathbf{Z}_{\underline{i}_n}(0)).$$

Thanks to Alexander's proof of wellposedness of the hard sphere dynamics [1], outside a set of zero measure the number of collisions is finite on any finite interval. Hence $[0,t]$ can be cut into small time intervals $[t_k,t_{k+1}]$ such that on each time interval there is at most one collision between two particles i and j, and if i (or j) is removed there is no collision at all. Using the semigroup property, one needs to prove the result only on each $[t_k,t_{k+1}]$.

We fix the number of particles $\mathcal{N}$ and the initial configuration $\mathbf{Z}_\mathcal{N}$ and we consider a small time t such that the preceding conditions are valid in $[0,t]$. We distinguish three cases.

First, suppose that on $[0,t]$ none of the particles in $\underline{i}_m$ collide. Then for any $n>m$ all the $\mathbb{1}_{\mathcal{R}^{m\leftarrow n,t}_{((s_i,\bar{s}_i)_i,(\kappa_j)_j)}}(\mathbf{Z}_{\underline{i}_n})$ vanish and

$$h_m\left(\mathbf{Z}_{\underline{i}_m}(t)\right) = \Phi^t_{m\leftarrow m}[h_m](\mathbf{Z}_{\underline{i}_n}(0)).$$

In the same way if a collision occurs between two particles of $\underline{i}_m$, the sets $\mathbb{1}_{\mathcal{R}^{m\leftarrow n,t}_{((s_i,\bar{s}_i)_i,(\kappa_j)_j)}}(\mathbf{Z}_{\underline{i}_n})$ vanish and the same equality holds.

Finally, suppose that the collision happens between one particle of $\underline{i}_m$ and another particle i_{m+1}. Up to a permutation of the indices, the collision happens between i_1 and i_{m+1}. Removing all the vanishing terms,

$$\begin{aligned}\sum_{n\geq m}\sum_{(i_{m+1},\cdots,i_n)}\Phi^t_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}) =& h_m\Big(\mathrm{Z}_m(t,\big(t,\big(()_0,(0)_m\big)\big)\mathbf{Z}_{\underline{i}_m})\Big)\\ &- h_m\Big(\mathrm{Z}_{m+1}\big(t,\big(((1,-1)),(0)_{m+1}\big),\mathbf{Z}_{\underline{i}_{m+1}}\big)\Big)\\ &+ h_m\Big(\mathrm{Z}_{m+1}\big(t,\big(((1,1)),(0)_{m+1}\big),\mathbf{Z}_{\underline{i}_{m+1}}\big)\Big).\end{aligned}$$

In the two first terms on the right hand side, particles move along straight lines because there is no scattering at the collision. Thus these two terms compensate. In the third term, since particles in $\underline{i}_{m+1}$ are deviated at the collision,

$$\mathrm{Z}_{m+1}\big(t,\big(((1,1)),(0)_{m+1}\big),\mathbf{Z}_{\underline{i}_{m+1}}\big)=\mathbf{Z}_{\underline{i}_m}(t)$$

and

$$\begin{aligned}h_m(\mathbf{Z}_{\underline{i}_m})(t)=h_m\Big(\mathrm{Z}_m(t,\big(t,\big((),(0)_m\big)\big)\mathbf{Z}_{\underline{i}_m})\Big)-h_m\Big(\mathrm{Z}_{m+1}\big(t,\big(((1,-1)),(0)_{m+1}\big),\mathbf{Z}_{\underline{i}_{m+1}}\big)\Big)\\ +h_m\Big(\mathrm{Z}_{m+1}\big(t,\big(((1,1)),(0)_{m+1}\big),\mathbf{Z}_{\underline{i}_{m+1}}\big)\Big)\ ,\end{aligned}$$

which concludes the proof. □

We conclude this section by observing that, applied to the covariance, the theorem gives the following formula:

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right]=\sum_{n\geq 1}\frac{1}{\mu_\varepsilon^{\frac12}}\mathbb{E}_\varepsilon\left[\sum_{(i_1,\cdots,i_n)}\Phi^t_{1\leftarrow n}[h](\mathbf{Z}_{\underline{i}_n}(0))\zeta^0_\varepsilon(g)\right].$$

As recalled in the introduction, since we do not know how take account the cancellations due to the factors $\bar{s}_i$, we can only bound $\Phi^t_{1\leftarrow n}[h]$ by means of an evaluation of the number of collision parameters. The problem is then that there no *a priori* bound on such number. To overcome this difficulty, we are led to introduce a conditioning of the invariant measure.

2.3. **Conditioning.** We shall need two conditionings of the initial data.
The first one is a symmetric conditioning on the full particle configuration.

Definition 2.2 (Distance cluster). Let L be a positive real number and $Z_n\in\mathbb{D}^n$ be a particle configuration. We consider the unoriented graph of vertices $\{1,\cdots,n\}$ and of edges

$$\{(i,j)\in[1,n]^2,\ d(x_i,x_j)<L\}.$$

A *L-distance cluster* is one of its connected component.

Let $\gamma>0$ be an integer depending only on the dimension, $\delta>0$ a time scale (which will be a power of ε) and $\mathbb{V}>0$ a velocity bound. In the following, we shall only look at $\gamma\delta\mathbb{V}$-distance clusters and we will therefore drop the "$\gamma\delta\mathbb{V}$".

We construct $\Upsilon_\varepsilon\subset\mathcal{D}_\varepsilon$ the set of particle configurations such that for any time $\tau\in\{0,\delta,2\delta,\cdots,t\}$, there is no distance cluster of size bigger than γ at time τ, and inside any subset of particles $\omega\subset[1,\mathcal{N}]$ with less than γ elements, $\frac12\|\mathbf{V}_\omega(\tau)\|^2$ is bounded by $\frac12\mathbb{V}^2$. We have the following bound on the measure of the complement of Υ_ε:

Proposition 2.3. *There exists a constant C_γ depending only on γ and on the dimension such that*

$$\mathbb{P}_\varepsilon\left(\Upsilon^c_\varepsilon\right)\leq C_\gamma\frac{t}{\delta}\left(\mu_\varepsilon\left(\mu_\varepsilon\delta^d\mathbb{V}^d\right)^\gamma+\mu_\varepsilon^\gamma e^{-\mathbb{V}^2/4}\right).\tag{2.5}$$

Proof.

$$\begin{aligned}\mathbb{P}_\varepsilon\Big(\Upsilon_\varepsilon^c\Big) &\le \sum_{k=0}^{t/\delta}\mathbb{E}_\varepsilon\left[\sum_{(i_1,\cdots,i_{\gamma+1})}\mathbb{1}_{\substack{\mathbf{X}_{\underline{i}_{\gamma+1}}(k\delta)\text{ form a}\\ \text{distance cluster}}}+\sum_{k'=1}^{\gamma}\sum_{(i_1,\cdots,i_{k'})}\mathbb{1}_{\|\mathbf{V}_{\underline{i}_{k'}}(k\delta)\|\ge\mathbb{V}}\right]\\
&\le \frac{t}{\delta}\left(\mu_\varepsilon^{\gamma+1}\int\mathbb{1}_{\substack{X_{\gamma+1}\text{ form a}\\ \text{distance cluster}}}M^{\otimes(\gamma+1)}dZ_{\gamma+1}+\sum_{k'=1}^{\gamma}\mu_\varepsilon^{\gamma}\int\mathbb{1}_{\|V_{k'}\|\ge\mathbb{V}}M^{\otimes(\gamma+1)}dZ_{\gamma+1}\right)\\
&\le C_\gamma\frac{t}{\delta}\left(\mu_\varepsilon^{\gamma+1}\left((\gamma\delta\mathbb{V})^d\right)^\gamma+\mu_\varepsilon^\gamma e^{-\frac{\mathbb{V}^2}{2}}\right)\end{aligned}$$

where C_γ is a constant depending only on the dimension and γ. We used that the Gibbs measure is time invariant and that the particles X_γ have to be at distance less than $\gamma\delta\mathbb{V}$ from $x_{\gamma+1}$ in order to form a distance cluster. ☐

Hence, if we set $\delta := \varepsilon^{1-\frac{1}{2d}}$, $\mathbb{V} := |\log\varepsilon|$ and fix $\gamma\in\mathbb{N}$ large enough, $\mathbb{P}_\varepsilon(\Upsilon_\varepsilon^c)$ is $O(\varepsilon^d)$.

The second conditioning is an asymmetric conditioning. We consider only a given, finite particle configuration Z_r. For fixed pseudotrajectory parameters $((s_i,\bar{s}_i)_{1\le i\le n-1},(\kappa_j)_{1\le j\le n})$, the configuration $Z_r\in\mathcal{D}_\varepsilon^r$ forms a *collision cluster* if the collision graph of $\mathrm{Z}_r(\tau,((s_i,\bar{s}_i)_i,(\kappa_j)_j),Z_r)$ on the time interval $[0,\delta]$ is connected. We define *local recollision* of $\mathrm{Z}_r(\tau)$ as the first collision (forward in time) which creates a cycle in the collision graph.

Definition 2.3. We define the function $\chi_r:\mathcal{D}_\varepsilon^r\mapsto\{0,1\}$ as the indicator function of:

$$\left\{Z_r\in\mathcal{D}_\varepsilon^r,\ \exists((s_i,\bar{s}_i)_{1\le i\le n-1},(\kappa_j)_{1\le j\le n}),\ \mathrm{Z}_r(\tau)\text{ forms a collision cluster with local recollision}\right\}.$$

We shall sometimes drop the index r from χ_r, when the cardinality of the cluster is clear from the context.

We have the following L^1 bound on χ_r:

Proposition 2.4. *There exists a positive constant C_r and some $\alpha>0$ depending only on the dimension such that*

$$\int_{\Lambda^{r-1}\times B_r(\mathbb{V})}\chi_r(Z_r)M^{\otimes r}(V_r)dX_{2,r}dV_r\le C_r\mu_\varepsilon^{-r+1}\delta^2\left(\mu_\varepsilon\delta^d\mathbb{V}^d\right)^{r-3}\varepsilon^\alpha \tag{2.6}$$

where $B_r(\mathbb{V})$ is the ball of radius $\mathbb{V}$ in dimension rd.

Proof. First of all we observe that, if the pseudotrajectories $\mathrm{Z}_r(\tau)$ form a collision cluster for some collision parameters, the initial positions in the configuration Z_r need to be close enough. As the speed of each particles is globally bounded by $\mathbb{V}$, there exists, for any couple of particle (i,i'), a finite sequence $i=j_1,\ j_2,\cdots,j_k=i'$ of two by two distinct indices such that

$$|x_{j_l}-x_{j_{l+1}}|\le 2\mathbb{V}\delta.$$

Thus, the distance between any two particles of Z_r is bounded by $2r\mathbb{V}\delta$. We need, however, a more precise geometric conditioning in order to obtain (2.6).

Let $Z_r\in\mathcal{D}_\varepsilon^r$ be such that $\chi(Z_r)$ is non zero. Then, there exists a set of pseudotrajectory parameters $((s_i,\bar{s}_i)_i,(\kappa_j)_j)$ such that the pseudotrajectory $\mathrm{Z}_r(\tau,((s_i,\bar{s}_i)_i,(\kappa_j)_j))$ has a local recollision. We define τ_s the time of such first local recollision. We construct a second set of recollision parameters:

$$\kappa_j'=\kappa_j(0)-\kappa_j(\tau_s).$$

Then for any $\tau \in [0,\tau_s]$,

$$\mathrm{Z}_r(\tau,((s_i,\bar{s}_i)_i,(\kappa_j)_j)) = \mathrm{Z}_r(\tau,((s_i,\bar{s}_i)_i,(\kappa'_j)_j))$$

and for all j, $\kappa'_j(\tau_s)=0$. Moreover, on $[0,\tau_s)$, the pseudotrajectory has no local recollision. Thus κ'_j is smaller than $r-1$.

Let $\varpi \subset [1,r]$ be the connected component of the collision graph $\mathcal{G}_r^{[0,\tau_s]}$ which contains the two particles involved in the local recollision. Because particles in ϖ do not interact with particles in ϖ^c, $\mathrm{Z}_r(t,Z_r)$ restricted to particles ϖ and to the time interval $[0,\tau_s]$ can be represented by a pseudotrajectory $\mathrm{Z}_{|\varpi|}(\tau,((s''_i,\bar{s}''_i)_i,(\kappa''_j)_j),Z_\varpi)$ for some collision parameters $((s''_i,\bar{s}''_i)_i,(\kappa''_j)_j)$. Note that we can take the κ''_j smaller than the maximum of the κ'_j and thus smaller than $r-1$. This gives a more precise constraint on Z_ϖ, and $\{Z_r,\, \chi_r(Z_r)=1\}$ is included in

$$\bigcup_{\varpi\subset\{1,\cdots,r\}}\ \bigcup_{\substack{(s_i,\bar{s}_i)_{i\leq|\varpi|}\\ (\kappa_j)\in[0,r-1]^{|\varpi|}}} \left\{ Z_r \,\middle|\, \begin{array}{c}\text{the collision graph of}\\ \mathrm{Z}_{|\varpi|}(\tau,((s_i,\bar{s}_i)_i,(\kappa_j)_j),Z_\varpi)\\ \text{is connected}\end{array} \right\}.$$

Note this union runs over a finite set of parameters.

We can now fix the pseudotrajectory parameters. The following standard estimation holds:

$$\int_{\Lambda^{|\varpi|-1}\times B_{|\varpi|}(\mathbb{V})} \mathbb{1}_{\substack{\mathrm{Z}_{|\varpi|}(\tau)\text{ forms a cluster}\\ \text{with local recollision}}} e^{-\frac{1}{2}\|V_r\|^2} dX_{\varpi\setminus\{\min\varpi\}} dV_\varpi \leq C_{|\varpi|}\mu_\varepsilon^{-|\varpi|+1}\delta^2\left(\mu_\varepsilon\delta^d\mathbb{V}^d\right)^{|\varpi|-3}\varepsilon^\alpha$$

where we used that $|\varpi|\geq 2$ and that $\varepsilon|\log\varepsilon|/\delta = O(\varepsilon^\alpha)$ for some α. It follows from the same proof than for Lemma 5.3 below, replacing in the bound the two times scales θ and t by δ (the unique time scale in the present case) and replacing the number of particle n'' by $|\varpi|$.

Using the distance constraints on Z_{ϖ^c} and summing on all possible parameters, we obtain the announced bound. $\square$

Finally, we denote $\mathcal{X}_{(i_1,\cdots,i_n)} : \{\mathbf{Z}_\mathcal{N}\in\mathcal{D}_\varepsilon,\ \mathcal{N}\geq\max\underline{i}_n\}\mapsto\{0,1\}$ the indicator function of the set

$$\left\{\mathbf{Z}_\mathcal{N}\in\mathcal{D}_\varepsilon \,\middle|\, \exists\varpi,\ \varpi\cap(i_1,\cdots,i_n)\neq\emptyset,\ \chi(\mathbf{Z}_\varpi)=1\right\}.$$

Note that $\mathcal{X}_{\underline{i}_n}$ depends on the 'background particles' different from $(i_1,\cdots,i_n)$.

Note also that we have:

$$\mathcal{X}_{(i_1,\cdots,i_n)}(\mathbf{Z}_\mathcal{N}) \ = \ 1 \quad - \prod_{\substack{\varpi\subset\{1,\cdots,\mathcal{N}\}\\ \varpi\cap\{i_1,\cdots,i_n\}\neq\emptyset}} \big(1-\chi(\mathbf{Z}_\varpi)\big). \tag{2.7}$$

The two conditionings introduced in this section allow us to bound the number of recollisions, as explained next. Let $Z_n\in\mathcal{D}_\varepsilon^n$ be an initial position such that there is no distance cluster of size bigger than γ (first conditioning) and for any $\sigma\subset\{1,\cdots,n\}$, $\chi(Z_\sigma)=0$ (second conditioning). Fix now collision parameters $((s_i,\bar{s}_i)_i,(\kappa_j)_j)$ such that the pseudotrajectory $\mathrm{Z}_n(t)$ has no recollision on $[\delta,t]$ (which will be ensured by a tuned sampling of collisions, introduced in the following section). Then due to the first, symmetric conditioning, any particle can only meet $\gamma-1$ different particles on $[0,\delta]$. Moreover due to the second, asymmetric conditioning, there is no local recollision on $[0,\delta]$. This implies that there are at most $\gamma-1$ recollisions per particle. In particular, any pseudotrajectory of this type can be parameterized by collision parameters

$$((s_i,\bar{s}_i)_{1\leq i\leq n-1},(\kappa_j)_{1\leq j\leq n})\in\{\pm1\}^{2(n-1)}\times[0,\gamma-1]^n.$$

2.4. **Sampling.** Let $\tau > 0$. Using the two conditionings of the previous section, for $\mathbf{Z}_{\mathcal{N}} \in \Upsilon_\varepsilon \cap \left\{\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}) = 0\right\}$ we have

$$h_m\left(\mathbf{Z}_{\underline{i}_m}(\tau)\right) = \sum_{n\geq m}\sum_{(i_{m+1},\cdots,i_n)} \Phi^{\gamma,\tau}_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)) \tag{2.8}$$

where

$$\Phi^{\gamma,\tau}_{m\leftarrow n}[h_m](Z_n) := \frac{1}{(n-m)!}\sum_{\substack{(s_i,\bar{s}_i)_{i\leq n-m}\\ (\kappa_j)_j\in[1,\gamma-1]^n}}\prod_{i=1}^{n-m}\bar{s}_i \mathbb{1}_{\mathcal{R}^{m\leftarrow n,\gamma,\tau}_{((s_i,\bar{s}_i),(\kappa_j))}} h_m(\mathrm{Z}_n(\tau))\,. \tag{2.9}$$

For generic t and particle configuration, we can then perform the following decomposition on Υ_ε

$$\begin{aligned}\sum_{\underline{i}_m} h_m\left(\mathbf{Z}_{\underline{i}_m}(t)\right) &= \sum_{\underline{i}_m} h_m(\mathbf{Z}_{\underline{i}_m}(t))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(t-\delta)) + \sum_{\underline{i}_m} h_m(\mathbf{Z}_{\underline{i}_m}(t))\big(1-\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(t-\delta))\big)\\ &= \sum_{\underline{i}_m} h_m(\mathbf{Z}_{\underline{i}_m}(t))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(t-\delta))\\ &\quad + \sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{\gamma,\delta}_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(t-\delta))\big(1-\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(t-\delta))\big)\,.\end{aligned} \tag{2.10}$$

Let $\Phi^{\gamma,\tau}_{m,n}[h_m]$ be the symmetrization $\Phi^{\gamma,\tau}_{m\leftarrow n}[h_m]$:

$$\Phi^{\gamma,\tau}_{m,n}[h_m](Z_n) := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\Phi^{\gamma}_{m\leftarrow n}[h_m](z_{\sigma(1)},\cdots,z_{\sigma(n)}).$$

There is a more explicit formula for such functional. We define $\mathcal{R}^{m,n,\gamma,\tau}_{((s_i,\bar{s}_i),(\kappa_j))} \subset \mathcal{D}^n_\varepsilon$ as the set of initial data such that pseudotrajectories $\mathrm{Z}_n(\cdot)$ have m remaining particles at time τ, $\kappa_j(\tau) = 0$ for all j and there is no local recollision. Then

$$\Phi^{\gamma,\tau}_{m,n}[h_m](Z_n) := \frac{1}{n!}\sum_{\substack{(s_i,\bar{s}_i)_{i\leq n-m}\\ (\kappa_j)_j,\in[0,\gamma-1]}}\prod_{i=1}^{n-m}\bar{s}_i\mathbb{1}_{\mathcal{R}^{m,n,\gamma,\tau}_{((s_i,\bar{s}_i),(\kappa_j))}} h_m(\mathrm{Z}_n(\tau)). \tag{2.11}$$

Finally we want to isolate pseudotrajectories with no recollision whatsoever: these will form the main contribution in the Boltzmann-Grad limit. We then define (i) the development along pseudotrajectories without recollision

$$\Phi^{0,\tau}_{m,n}[h_m](Z_n) := \frac{1}{n!}\sum_{(s_i,\bar{s}_i)_{i\leq n-m}}\prod_{i=1}^{n-m}\bar{s}_i\mathbb{1}_{\mathcal{R}^{m,n,\tau}_{(s_i,\bar{s}_i)}} h_m(\mathrm{Z}_n(\tau)), \tag{2.12}$$

where $\mathcal{R}^{m,n,\tau}_{(s_i,\bar{s}_i)} \subset \mathcal{R}^{m,n,\gamma,\tau}_{(s_i,\bar{s}_i),(0)_j}$ is such that the pseudotrajectories have no recollision, and (ii) the development along pseudotrajectories with non-pathological recollisions

$$\Phi^{>,\tau}_{m,n}[h_m] := \Phi^{\gamma,\tau}_{m,n}[h_m] - \Phi^{0,\tau}_{m,n}[h_m]. \tag{2.13}$$

We bring together all these decompositions and obtain, on Υ_ε,

$$\begin{aligned}\sum_{\underline{i}_m} h_m\left(\mathbf{Z}_{\underline{i}_m}(\tau)\right) &= \sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{0,\delta}_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(\tau-\delta)) + \sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{>,\delta}_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(\tau-\delta))\\ &+\sum_{\underline{i}_m} h_m(\mathbf{Z}_{\underline{i}_m}(\tau))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(\tau-\delta))\\ &-\sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{\gamma,\delta}_{m\leftarrow n}[h_m](\mathbf{Z}_{\underline{i}_n}(\tau-\delta))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(\tau-\delta)).\end{aligned}\tag{2.14}$$

The first term is an expansion along pseudotrajectories with no recollision. It is the main part of the sum. The rest takes into account the recollisions in the hard sphere dynamics.

We iterate this decomposition:

$$\begin{aligned}\sum_{\underline{i}_m} h_m\left(\mathbf{Z}_{\underline{i}_m}(\tau)\right) = &\sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{0,\tau}_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(0)) + \sum_{k=0}^{\tau/\delta}\sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{>,k\delta}_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(\tau-k\delta))\\ &+\sum_{n\geq m}\sum_{\underline{i}_n}\Phi^{0,(k-1)\delta}_{m,n}[h_m](\mathbf{Z}_{\underline{i}_n}(\tau-(k-1)\delta))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(\tau-k\delta))\\ &-\sum_{n'\geq n\geq m}\sum_{\underline{i}_{n'}}\Phi^{\gamma,\delta}_{n\leftarrow n'}\left[\Phi^{0,(k-1)\delta}_{m,n}[h_m]\right](\mathbf{Z}_{\underline{i}_{n'}}(\tau-k\delta))\mathcal{X}_{\underline{i}_m}(\mathbf{Z}_{\mathcal{N}}(\tau-k\delta)).\end{aligned}$$

The final ingredient is now a second sampling on a longer time scale $\theta = 1/\beta\log|\log\varepsilon|$ controlling the growth of the number of collisions. We denote $K := t/\theta \in \mathbb{N}$ and $K' := \theta/\delta \in \mathbb{N}$.

We obtain the following decomposition

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] = G^{\mathrm{main}}_\varepsilon(t) + G^{\mathrm{clust}}_\varepsilon(t) + G^{\mathrm{exp}}_\varepsilon(t) + G^{\mathrm{rec},1}_\varepsilon(t) + G^{\mathrm{rec},2}_\varepsilon(t)\tag{2.15}$$

with $G^{\mathrm{main}}_\varepsilon(t)$ the main part:

$$G^{\mathrm{main}}_\varepsilon(t) := \sum_{\substack{\underline{n}:=(n_j)_{j\leq K}\\ 0<n_j-n_{j-1}\leq 2^j}}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_{n_K}}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\right]\tag{2.16}$$

where

$$\Phi^0_{\underline{n}}[h] := \Phi^{0,\theta}_{n_{K-1},n_K}\circ\Phi^{0,\theta}_{n_{K-2},n_{K-1}}\cdots\circ\Phi^{0,\theta}_{1,n_1}[h]$$

is the development of h along pseudotrajectories with n_k annihilations on the time interval $[t-(k-1)\theta, t-k\theta]$ and no recollision, denoting $\underline{n} = (n_1,\cdots,n_K)$; moreover:

$$G^{\mathrm{clust}}_\varepsilon(t) := \mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right] - \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_{n_K}}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right]\tag{2.17}$$

corresponding to the symmetric conditioning;

$$G^{\mathrm{exp}}_\varepsilon(t) := \sum_{k=1}^K\sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}}\sum_{n_k>2^k+n_{k-1}}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_{n_k}}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\tag{2.18}$$

corresponding with trees with superexponential growth;

$$G_\varepsilon^{\mathrm{rec},1}(t) := \sum_{\substack{1\le k\le K-1\\ 1\le k'\le K'}} \sum_{\substack{n_1\le\cdots\le n_k\\ n_j-n_{j-1}\le 2^j}} \sum_{n''\ge n'\ge n_k} \mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_{n''}} \Phi^{>,k'}_{\underline{n},n',n''}[h]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right] \tag{2.19}$$

corresponding to (non local) recollisions; and finally $G_\varepsilon^{\mathrm{rec},2}(t)$:

$$\begin{aligned}&\sum_{\substack{1\le k\le K-1\\ 1\le k'\le K'}} \sum_{\substack{n_1\le\cdots\le n_k\\ n_j-n_{j-1}\le 2^j}} \left(\sum_{n'\ge n_k}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n'})}\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t_s+\delta)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right.\\ &\qquad\left. - \sum_{n''\ge n'\ge n_k}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n''})}\Phi^{\gamma,\delta}_{n'\leftarrow n''}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right)\end{aligned} \tag{2.20}$$

corresponding to pathological pseudotrajectories. In the last two terms, we have denoted $t_s := t-(k-1)\theta - k'\delta$ the *stopping time*, and

- $\Phi^{0,k'}_{\underline{n},n'}[h] := \Phi^{0,k'\delta}_{n_k,n'}\circ\Phi^0_{\underline{n}}[h]$, the pseudotrajectory development with no recollision, n' annihilations on $[0,(k'-1)\delta]$ and for $j<k$, n_j annihilations on $[(k'-1)\delta+(k-j)\theta,(k'-1)\delta+(k-j+1)\theta]$,
- $\Phi^{>,k'}_{\underline{n},n',n''}[h] := \Phi^{>,\delta}_{n',n''}\circ\Phi^{0,k'}_{\underline{n},n'}[h]$, the pseudotrajectory development with no recollision on $[\delta,k'\delta+k\theta]$, n'' annihilations on $[0,\delta]$, n' annihilations on $[\delta,k'\delta]$ and for $j<k$ and n_j annihilations on $[k'\delta+(k-j)\theta,k'\delta+(k-j-1)\theta]$, and at the least one recollision.

We stress that, thanks to the conditioning, each pseudotrajectory appearing in this representation has at most γ recollisions per particle.

3. Quasi-orthogonality estimates

The different error terms obtained in the previous section are of the form

$$\mathbb{E}_\varepsilon\left[\sum_{\underline{i}_n}\Phi_n[h](\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]$$

with $\Phi_n : L^\infty(\mathbb{D})\to L^\infty(\mathbb{D}^n)$ some continuous functional. In order to bound the errors, we will need an $L^2(\mathbb{P}_\varepsilon)$ bound on $\sum_{\underline{i}_n}\Phi_n[h](\mathbf{Z}_{\underline{i}_n})$. Such bound is derived in the following sections from detailed estimations on the functionals $\Phi_n[h]$. We will use, in particular, that we can bound the $\Phi_n[h](Z_n)$ by looking only at the relative positions of particles inside Z_n.

In the following we denote for $y\in\Lambda$

$$\tau_y : \left\{\begin{array}{ccc}\mathbb{D}^n & \to & \mathbb{D}^n\\ (X_n,V_n) & \mapsto & (x_1+y,\cdots,x_n+y,V_n).\end{array}\right. \tag{3.1}$$

Theorem 3.1. *Fix $m<n$ two positive integers, and g_n, h_m two functions on $\mathbb{D}^n$ and $\mathbb{D}^m$ such that there exists a finite sequence $(c_0,c_0',c_1,\cdots,c_n)\in\mathbb{R}_+^{n+2}$ bounding g_n, h_m in the following way:*

$$\int_{x_1=0}\sup_{y\in\Lambda}\left|g_n\left(\tau_y Z_n\right)\right|M^{\otimes n}(V_n)dX_{2,n}dV_n \le c_0, \tag{3.2}$$

$$\int_{x_1=0} \sup_{y\in\Lambda} \big|h_m\big(\tau_y Z_m\big)\big| M^{\otimes m}(V_m) dX_{2,m} dV_m \leq c_0' \tag{3.3}$$

and for all $l \in [1, m]$

$$\int_{x_1=0} \sup_{y\in\Lambda} \big|g_n\big(\tau_y Z_n\big) h_m\big(\tau_y Z_{n+1-l,n+m-l}\big)\big| M^{\otimes(n+m-l)}(V_{n+m-l}) dX_{2,n+m-l} dV_{n+m-l} \leq \frac{\mu_\varepsilon^{l-1}}{n^l} c_l. \tag{3.4}$$

There exists a constant $C > 0$ *depending only on the dimension such that*

$$\big|\mathbb{E}_\varepsilon\big[g_n\big]\big| \leq C^n c_0, \;\; \big|\mathbb{E}_\varepsilon\big[h_m\big]\big| \leq C^m c_0' \tag{3.5}$$

and denoting

$$g_n \circledast_l h_m(Z_{n+m-l}) = \frac{1}{(n+m-l)!} \sum_{\sigma\in\mathfrak{S}_{n+m-l}} g_n(Z_{\sigma([1,n])}) h_m(Z_{\sigma([n+1-l,n+m-l])}), \tag{3.6}$$

$$\mathbb{E}_\varepsilon\Big[\mu_\varepsilon \hat{g}_n \hat{h}_m\Big] = \sum_{l=1}^m \binom{n}{l}\binom{m}{l} \frac{l!}{\mu_\varepsilon^{l-1}} \mathbb{E}_\varepsilon\big[g_n \circledast_l h_m\big] + O\big(C^{n+m} c_0 c_0' \varepsilon\big). \tag{3.7}$$

In particular

$$|\mathbb{E}_\varepsilon\big[\mu_\varepsilon \hat{g}_n \hat{h}_m\big]| \leq C^{n+m} \sum_{l=1}^m c_l + C^{n+m} c_0 c_0' \,\varepsilon. \tag{3.8}$$

Proof of Theorem 3.1.

- We begin by the proof of (3.5).

Using invariance under permutation,

$$\begin{aligned}
\mathbb{E}_\varepsilon[g_n] &= \frac{1}{\mu_\varepsilon^n \mathcal{Z}_\varepsilon} \sum_{p\geq n} \frac{\mu_\varepsilon^p}{p!} \int \sum_{\substack{(i_1,\cdots i_n)\\ \forall k, i_k\leq p}} g_n(Z_n) e^{-\mathcal{H}_p^\varepsilon(Z_p)} \frac{dZ_p}{(2\pi)^{dp/2}} \\
&= \frac{1}{\mu_\varepsilon^n \mathcal{Z}_\varepsilon} \sum_{p\geq n} \frac{\mu_\varepsilon^p}{p!} \frac{p!}{(n-p!)} \int g_n(Z_n) e^{-\mathcal{H}_p^\varepsilon(Z_p)} \frac{dZ_p}{(2\pi)^{dp/2}} \\
&= \frac{1}{\mathcal{Z}_\varepsilon} \sum_{p\geq 0} \frac{\mu_\varepsilon^p}{p!} \int g_n(Z_n) e^{-\mathcal{V}_{n+p}^\varepsilon(X_n, \underline{X}_p)} M^{\otimes n} dZ_n d\underline{X}_p.
\end{aligned}$$

We denote in the following $\Omega := \{X_n, \underline{x}_1, \cdots, \underline{x}_p\}$ and for $X, Y \in \Omega$,

$$\varphi(X,Y) := -\mathbb{1}_{d(X,Y)\leq\varepsilon}$$

and we decompose $\exp\big(-\mathcal{V}_{n+p}^\varepsilon(X_{n+1}, \underline{X}_p)\big)$

$$e^{-\mathcal{V}_{n+p}^\varepsilon(X_{n+1},\underline{X}_p)} = e^{-\mathcal{V}_n^\varepsilon(X_n)} \prod_{\substack{(X,Y)\in\Omega^2\\ X\neq Y}} (1+\varphi(X,Y)) = e^{-\mathcal{V}_n^\varepsilon(X_n)} \sum_{G\in\mathcal{G}(\Omega)} \prod_{(X,Y)\in E(G)} \varphi(X,Y)$$

where $\mathcal{G}$ is the set of non oriented graphs on Ω and $E(G)$ the set of edges of G. Denoting by $\mathcal{C}(\omega)$ the set of connected graphs on ω,

$$\begin{aligned}
&\exp\big(-\mathcal{V}^\varepsilon_{n+p}(X_{n+1},\underline{X}_p)\big)\\
&=\sum_{\omega\subset[1,p]}\left(e^{-\mathcal{V}^\varepsilon_n(X_n)}\sum_{G\in\mathcal{C}(\omega\cup\{X_n\})}\prod_{(X,Y)\in E(G)}\varphi(X,Y)\sum_{G\in\mathcal{G}(\omega^c)}\prod_{(X,Y)\in E(G)}\varphi(X,Y)\right)\\
&=\sum_{\omega\subset[1,p]}\left(e^{-\mathcal{V}^\varepsilon_n(X_n)-\mathcal{V}^\varepsilon_{|\omega^c|}(\underline{X}_{\omega^c})}\sum_{G\in\mathcal{C}(\omega\cup\{X_n\})}\prod_{(X,Y)\in E(G)}\varphi(X,Y)\right)\\
&=:\sum_{\omega\subset[1,p]}e^{-\mathcal{V}^\varepsilon_{|\omega^c|}(\underline{X}_{\omega^c})}\psi^n_p(X_n,\underline{X}_\omega)\,.
\end{aligned}\tag{3.9}$$

Thus, using exchangeability,

$$\begin{aligned}
\mathbb{E}_\varepsilon[g_n]&=\frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq0}\sum_{p_1+p_2=p}\frac{\mu_\varepsilon^p}{p!}\frac{p!}{p_1!p_2!}\int g_n(Z_n)\psi^n_{p_1}(X_n,\underline{X}_{p_1})e^{-\mathcal{V}^\varepsilon_{p_2}(\underline{X}'_{p_2})}M^{\otimes n}dZ_nd\underline{X}_{p_1}d\underline{X}'_{p_2}\\
&=\left(\frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq0}\frac{\mu_\varepsilon^p}{p!}\int e^{-\mathcal{V}^\varepsilon_p(\underline{X}_p)}d\underline{X}_p\right)\left(\sum_{p\geq0}\frac{\mu_\varepsilon^p}{p!}\int g_n(Z_n)\psi^n_p(X_n,\underline{X}_p)M^{\otimes n}dZ_nd\underline{X}_p\right)\\
&=\sum_{p\geq0}\frac{\mu_\varepsilon^p}{p!}\int g_n(Z_n)\psi^n_p(X_n,\underline{X}_p)M^{\otimes n}dZ_nd\underline{X}_p.
\end{aligned}\tag{3.10}$$

We recall Penrose tree inequality (see [19, 6, 16]),

$$\left|\sum_{C\in\mathcal{C}(\Omega)}\prod_{(X,Y)\in E(C)}\varphi(X,Y)\right|\leq\sum_{T\in\mathcal{T}(\Omega)}\prod_{(X,Y)\in E(T)}|\varphi(X,Y)|\tag{3.11}$$

with $\mathcal{T}(\Omega)$ the set of trees (minimally connected graphs) on Ω. Fix $\tau_{-x_1}X_n$ (the relative position between particles). Integrating a constraint $\varphi(\underline{x}_i,\underline{x}_j)$ provides a factor $\mathbf{c}_d\varepsilon^d$, $\varphi(X_n,\underline{x}_j)$ a factor $n\mathbf{c}_d\varepsilon^d$ (where $\mathbf{c}_d$ is the volume of a sphere of diameter 1). As there are

$$\frac{(p-1)!}{(d_0-1)!(d_1-1)!\cdots(d_p-1)!}$$

trees with specified vertex degrees $d_0,\cdots,d_p$ associated to vertices X_n, $\underline{x}_1,\cdots,\underline{x}_p$ (see [16, 6]), we get

$$\begin{aligned}
\left|\int\psi^n_p(X_n\underline{X}_p)d\underline{X}_pdx_1\right|&\leq\sum_{\substack{d_1,\cdots,d_p\geq1\\d_0+\cdots+d_p=2p}}\frac{(p-1)!}{(d_0-1)!(d_1-1)!\cdots(d_p-1)!}n^{d_0}(\mathbf{c_d}\varepsilon^d)^p\\
&\leq(p-1)!(\mathbf{c_d}\varepsilon^d)^p\left(\sum_{d_0\geq1}\frac{n^{d_0}}{(d_0-1)!}\right)\left(\sum_{d_1\geq1}\frac{1}{(d_1-1)!}\right)\cdots\left(\sum_{d_p\geq1}\frac{1}{(d_p-1)!}\right)\\
&\leq(p-1)!\big(\mathbf{c_d}\varepsilon^d\big)^pne^{n+p}.
\end{aligned}\tag{3.12}$$

We can integrate on the rest of parameters using (3.2). Hence

$$|\mathbb{E}_\varepsilon[g_n]| \leq \sum_{p\geq 0} \frac{(p-1)!\big(\mathbf{c_d} e\varepsilon^d\big)^p n e^n}{p!} \int |g_n(Z_n)| e^{-\frac{\|V_n\|^2}{2}} \frac{dZ_n}{(2\pi)^{dn/2}} \leq c_0 \sum_{p\geq 0} C^n (C\varepsilon)^p$$

which converges for ε small enough. This concludes the proof of (3.5).

- We treat now (3.7). Recall first that

$$\mathbb{E}_\varepsilon\Big[\mu_\varepsilon \hat{g}_n \hat{h}_m\Big] = \frac{1}{\mu_\varepsilon^{n+m-1}} \mathbb{E}_\varepsilon\left[\sum_{\underline{i}_n} g_n(\mathbf{Z}_{\underline{i}_n}) \sum_{\underline{j}_m} h_m(\mathbf{Z}_{\underline{j}_m})\right] - \mu_\varepsilon \mathbb{E}_\varepsilon\left[g_n\right] \mathbb{E}_\varepsilon\left[h_m\right].$$

Let us count the number of ways such that $\underline{i}_n$ and $\underline{j}_m$ can intersect on a set of length l. We have to choose two sets $A \subset [n]$ and $A' \subset [m]$ of length l, and a bijection $\sigma : A \to A'$ such that for all indices $k \in A$, $i_k = j_{\sigma k}$ and that $\underline{i}_{A^c}$ does not intersect $\underline{j}_{(A^c)'}$. Thus using the symmetry,

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\mu_\varepsilon \hat{g}_n \hat{h}_m\Big] =& \sum_{l=1}^m \binom{n}{l}\binom{m}{l} \frac{l!}{\mu_\varepsilon^{l-1}} \mathbb{E}_\varepsilon\big[g_n \circledast_l h_m\big] \\ &+ \mu_\varepsilon \left(\mathbb{E}_\varepsilon\left[\frac{1}{\mu_\varepsilon^{n+m}} \sum_{\underline{i}_{n+m}} g_n(\mathbf{Z}_{\underline{i}_n}) h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}})\right] - \mathbb{E}_\varepsilon\left[g_n\right] \mathbb{E}_\varepsilon\left[g\right]\right).\end{aligned}$$

To estimate the error term in the second line, we write

$$\begin{aligned}&\mathbb{E}_\varepsilon\left[\frac{1}{\mu_\varepsilon^{n+m}} \sum_{\underline{i}_{n+m}} g_n(\mathbf{Z}_{\underline{i}_n}) h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}})\right] \\ &\quad = \frac{1}{\mathcal{Z}_\varepsilon} \sum_{p\geq 0} \frac{\mu_\varepsilon^p}{p!} \int g_n(Z_n) h_m(Z'_m) \exp\big(-\mathcal{V}^\varepsilon_{n+m+p}(X_n, X'_m, \underline{X}_p)\big) M^{\otimes n} dZ_n M^{\otimes m} dZ'_m d\underline{X}_p.\end{aligned}$$

We denote in the following $\Omega := \{X_n, X'_m, \underline{x}_1, \cdots, \underline{x}_p\}$ and we have that

$$\begin{aligned}\exp\big(-\mathcal{V}^\varepsilon_{n+m+p}(X_n, X'_m, \underline{X}_p)\big) &= e^{-\mathcal{V}^\varepsilon_n(X_n)} e^{-\mathcal{V}^\varepsilon_m(X'_m)} \prod_{\substack{(X,Y)\in\Omega^2 \\ X\neq Y}} (1 + \varphi(X,Y)) \\ &= e^{-\mathcal{V}^\varepsilon_n(X_n)} e^{-\mathcal{V}^\varepsilon_m(X'_m)} \sum_{G\in\mathcal{G}(\Omega)} \prod_{(X,Y)\in E(G)} \varphi(X,Y)\,.\end{aligned}$$

Partitioning on the connected components of X_n and X'_m,

$$\begin{aligned}
&\exp\ \big(-\mathcal{V}^\varepsilon_{n+m+p}(X_n, X'_m, \underline{X}_p)\big)\\
&\quad= \sum_{\omega\subset[1,p]} \Bigg(\exp\Big(-\mathcal{V}^\varepsilon_n(X_n) - \mathcal{V}^\varepsilon_m(X'_m) - \mathcal{V}^\varepsilon_{|\omega^c|}(\underline{X}_{\omega^c})\Big) \sum_{\substack{G\in\mathcal{C}(\omega\cup\\ \{X_n,X'_m\})}} \prod_{(X,Y)\in E(G)} \varphi(X,Y)\Bigg)\\
&\qquad\qquad + \sum_{\substack{\omega_1,\omega_1\subset[1,p]\\ \omega_1\cap\omega_2=\emptyset}} \psi^n_{|\omega_1|}(X_n,\underline{X}_{\omega_1})\psi^m_{|\omega_2|}(X'_m,\underline{X}_{\omega_2}) e^{-\mathcal{V}^\varepsilon_{|(\omega_1\cup\omega_2)^c|}(\underline{X}_{(\omega_1\cup\omega_2)^c})}\\
&\quad=: \sum_{\omega\subset[1,p]} \psi^{n,m}_{|\omega|}(X_n, X'_m, \underline{X}_\omega) e^{-\mathcal{V}^\varepsilon_{||\omega^c|}(\underline{X}_{\omega^c})}\\
&\qquad\qquad + \sum_{\substack{\omega_1,\omega_1\subset[1,p]\\ \omega_1\cap\omega_2=\emptyset}} \psi^n_{|\omega_1|}(X_n,\underline{X}_{\omega_1})\psi^m_{|\omega_2|}(X'_m,\underline{X}_{\omega_2}) e^{-\mathcal{V}^\varepsilon_{|(\omega_1\cup\omega_2)^c|}(\underline{X}_{(\omega_1\cup\omega_2)^c})}.
\end{aligned}$$

Using the invariance under permutation and (3.10)

$$\begin{aligned}
&\frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq 0}\frac{\mu_\varepsilon^p}{p!}\int g_n(Z_n)h_m(Z'_m) \sum_{\substack{\omega_1,\omega_1\subset[1,p]\\ \omega_1\cap\omega_2=\emptyset}} \psi^n_{|\omega_1|}(X_n,\underline{X}_{\omega_1})\psi^m_{|\omega_2|}(X'_m,\underline{X}_{\omega_2}) e^{-\mathcal{V}^\varepsilon_{|(\omega_1\cup\omega_2)^c|}(\underline{X}_{(\omega_1\cup\omega_2)^c})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times M^{\otimes(n+m)} dZ_n dZ'_m M^{\otimes n} dZ_n M^{\otimes n'} dZ'_{n'} d\underline{X}_p\\
&\quad= \frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq 0}\sum_{p_1+p_2+p_3=p}\frac{\mu_\varepsilon^p}{p!}\frac{p!}{p_1!p_2!p_3!}\int g_n(Z_n)h_{n'}(Z'_{n'})\psi^n_{p_1}(X_n,\underline{X}_{p_1})\psi^1_{p_2}(x_{n+1},\underline{X}'_{p_2})\\
&\qquad\qquad\qquad\qquad \times\Big(M^{\otimes n}dZ_n d\underline{X}_{p_1}\Big)\Big(M^{\otimes n'}dZ'_{n'}d\underline{X}'_{p_2}\Big)\Big(e^{-\mathcal{V}^\varepsilon_{p_3}(\underline{X}''_{p_3})}d\underline{X}''_{p_3}\Big)\\
&\quad= \mathbb{E}_\varepsilon[g_n]\mathbb{E}_\varepsilon[h_{n'}],
\end{aligned}$$

and in the same way

$$\begin{aligned}
&\frac{1}{\mathcal{Z}_\varepsilon}\ \sum_{p\geq 0}\frac{\mu_\varepsilon^p}{p!}\int g_n(Z_n)h_m(Z'_m)\sum_{\omega\subset[1,p]}\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_\omega)e^{-\mathcal{V}^\varepsilon_{||\omega^c|}(\underline{X}_{\omega^c})}M^{\otimes n}dZ_nM^{\otimes m}dZ'_m d\underline{X}_p\\
&\quad= \frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq 0}\sum_{p_1+p_2=p}\frac{\mu_\varepsilon^p}{p!}\frac{p!}{p_1!p_2!}\int g_n(Z_n)h_m(Z'_m)\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad e^{-\mathcal{V}^\varepsilon_{p_2}(\underline{X}'_{p_2})}M^{\otimes(n+m)}dZ_ndZ'_m d\underline{X}_{p_1}d\underline{X}'_{p_2}\\
&\quad= \sum_{p_1\geq 0}\frac{\mu_\varepsilon^p}{p_1!}\int g_n(Z_n)h_m(Z'_m)\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})M^{\otimes(n+m)}dZ_ndZ'_m d\underline{X}_{p_1}d\underline{X}'_{p_2}.
\end{aligned}$$

Using again Penrose tree inequality,

$$\left|\psi^{n,m}_{|\omega|}(X_n,X'_m,\underline{X}_{p_1})\right| \leq \sum_{T\in\mathcal{T}(\Omega)}\prod_{(X,Y)\in E(T)}|\varphi(X,Y)|\,. \tag{3.13}$$

Fix $\tau_{-x_1}X_n$ and $\tau_{-x'_1}X'_m$. Integrating a constraint $\varphi(\underline{x}_i,\underline{x}_j)$ provides a factor $\mathbf{c}_d\varepsilon^d$, $\varphi(X_n,\underline{x}_j)$ a factor $n\mathbf{c}_d\varepsilon^d$, $\varphi(X'_m,\underline{x}_j)$ a factor $m\mathbf{c}_d\varepsilon^d$ and $\varphi(X_n,X'_m)$ a factor $nm\mathbf{c}_d\varepsilon^d$. Denoting $d_0, d'_0, d_1\cdots, d_p$

the degrees of X_n, X'_m, $\underline{x}_1, \cdots, \underline{x}_m$ we get

$$\begin{aligned}
&\left| \int \psi^{n,m}_{|\omega|}(X_n, X'_m, \underline{X}_{p_1}) d\underline{X}_p dx_1 dx'_1 \right| \\
&\qquad \leq \sum_{\substack{d'_0, d_0, \cdots, d_p \geq 1 \\ d'_0 + d_0 + \cdots + d_p = 2p}} \frac{p!}{(d'_0 - 1)(d_0 - 1)! \cdots (d_p - 1)!} n^{d_0} m^{d'_0} (\mathbf{c_d} \varepsilon^d)^{+1} \\
&\qquad \leq p! \big(\mathbf{c_d} \varepsilon^d\big)^{p+1} nm \, e^{n+m+p}.
\end{aligned} \tag{3.14}$$

We can integrate on the rest of parameters using (3.2) and (3.3), and finally

$$\begin{aligned}
&\mu_\varepsilon \left(\mathbb{E}_\varepsilon \left[\frac{1}{\mu_\varepsilon^{n+m}} \sum_{\underline{i}_{n+m}} g_n(\mathbf{Z}_{\underline{i}_n}) h_m(\mathbf{Z}_{\underline{i}_{n+1,n+m}}) \right] - \mathbb{E}_\varepsilon \left[g_n \right] \mathbb{E}_\varepsilon \left[g \right] \right). \\
&\leq c_0 c'_0 \mu_\varepsilon \sum_{p \geq 0} \frac{\mu_\varepsilon^p}{p!} p! \big(\mathbf{c_d} \varepsilon^d\big)^{p+1} nm \, e^{n+m+p} \\
&\leq \mu_\varepsilon \varepsilon^d nm (\mathbf{c_d} e)^{n+m} c_0 c'_0 \sum_{p \geq 0} (e \mathbf{c_d} \varepsilon)^p \\
&\leq \varepsilon C^{n+m+1} \sum_{p \geq 0} (e \mathbf{c_d} \varepsilon)^p
\end{aligned}$$

which converges for ε small enough. $\square$

Note also the following bound in L^p norms of the fluctuation field.

Theorem 3.2. *For any $p \in [2, \infty)$, there exists a constant $C_p > 0$ such that*

$$\big(\mathbb{E}_\varepsilon \left[\zeta^0_\varepsilon(g)^p \right]\big)^{1/p} \leq C_p \|g\|_{L^p(M(v)dz)}. \tag{3.15}$$

The proof can be found in Appendix A of [7].

From these estimations we can deduce the following corollary:

Corollary 3.3. *Let h_n be a test function satisfying the conditions of Theorem 3.1. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\left| \mathbb{E}_\varepsilon \left[\mu_\varepsilon^{-1/2} \sum_{(i_1, \cdots, i_n)} h_n(\mathbf{Z}_{\underline{i}_n}(t_s)) \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] \right| \\
&\qquad\qquad \leq C^n \mu_e^{n-1} \mathbb{E}_\varepsilon \left[\zeta^0_\varepsilon(g)^2 \right]^{1/2} \left(c_0 + \left(\sum_{l=1}^n c_l \right)^{1/2} \right).
\end{aligned} \tag{3.16}$$

Proof.

$$\begin{aligned}
\mathbb{E}_\varepsilon \left[\mu_\varepsilon^{-1/2} \sum_{(i_1, \cdots, i_n)} h_n(\mathbf{Z}_{\underline{i}_n}(t_s)) \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] &= \mu_\varepsilon^{n-1} \mathbb{E}_\varepsilon \left[\mu_\varepsilon^{1/2-n} \sum_{(i_1, \cdots, i_n)} h_n(\mathbf{Z}_{\underline{i}_n}(t_s)) \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] \\
&= \mu_\varepsilon^{n-1} \left(\mathbb{E}_\varepsilon \left[\mu_\varepsilon^{1/2} \widehat{h_n}(\mathbf{Z}_{\mathcal{N}}(t_s)) \, \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] + \mathbb{E}_\varepsilon \left[h_n \right] \mathbb{E}_\varepsilon \left[\mu_\varepsilon^{1/2} \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] \right) \\
&= \mu_\varepsilon^{n-1} \left(\mathbb{E}_\varepsilon \left[\mu_\varepsilon^{1/2} \widehat{h_n}(\mathbf{Z}_{\mathcal{N}}(t_s)) \, \zeta^0_\varepsilon(g) \mathbb{1}_{\Upsilon_\varepsilon} \right] + \mathbb{E}_\varepsilon \left[h_n \right] \mathbb{E}_\varepsilon \left[\zeta^0_\varepsilon(g) \mu_\varepsilon^{1/2} \left(1 - \mathbb{1}_{\Upsilon^c_\varepsilon} \right) \right] \right).
\end{aligned}$$

By $\mathbb{E}_\varepsilon[\zeta^0_\varepsilon(g)] = 0$ and using Cauchy-Schwartz inequality, we find

$$\left|\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_n)} h_n(\mathbf{Z}_{\underline{i}_n}(t_s))\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right|$$
$$\leq \mu_\varepsilon^{n-1}\left(\mathbb{E}_\varepsilon\left[\mu_\varepsilon\left[\widehat{h_n}\right]^2\right]^{\frac12}\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac12} + \mathbb{E}_\varepsilon\left[h_n\right]\mathbb{E}_\varepsilon\left[\zeta^0_\varepsilon(g)^2\right]^{\frac12}\left(\mu_\varepsilon\mathbb{P}_\varepsilon\left[\Upsilon^c_\varepsilon\right]\right)^{\frac12}\right).$$

We apply now Theorem 3.1. The bound on $\mathbb{P}_\varepsilon\left[\Upsilon^c_\varepsilon\right]$ given in section 2.3 and the bound on the L^p norm of $\zeta^0_\varepsilon(g)$ (3.15) lead to the stated corollary. □

4. Clustering estimations

The objective of this section is to bound $G^{\text{clust}}_\varepsilon(t)$ and $G^{\text{exp}}_\varepsilon(t)$, defined by

$$G^{\text{clust}}_\varepsilon(t) := \mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right] - \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n_K})}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right],$$

$$G^{\text{exp}}_\varepsilon(t) := \sum_{k=1}^{K}\sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}}\sum_{n_k>2^k+n_{k-1}}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n_k})}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right].$$

Proposition 4.1. *For $\varepsilon>0$ small enough,*

$$\left|G^{exp}_\varepsilon(t) + G^{clust}_\varepsilon(t)\right| \leq C\|g\|\,\|h\|\,\left(\varepsilon^{1/2}(Ct)^{2^{t/\theta}} + \ t\theta^{1/2}\right) \tag{4.1}$$

To obtain the stated result, we need first the following bounds on the pseudotrajectory developments without recollisions of type $\Phi^0_{\underline{n}}[h]$:

Proposition 4.2. *Fix $k\in\mathbb{N}$ and $\underline{n} := (n_1,\cdots,n_k)\in\mathbb{N}^k$ with $n_1\leq n_2\leq\cdots\leq n_k$. Then*

$$\int_{x_1=0}\sup_{y\in\Lambda}\left|\Phi^0_{\underline{n}}[h](\tau_y Z_{n_k})\right|M^{\otimes n_k}dV_{n_k}dX_{2,n_k} \leq \frac{\|h\|}{\mu_\varepsilon^{n_k-1}}C^{n_k}\theta^{n_k-n_{k-1}}t^{n_{k-1}-1}, \tag{4.2}$$

and, for $m\in[1,n_k]$,

$$\begin{aligned}&\int_{x_1=0}\sup_{y\in\Lambda}\left|\Phi^0_{\underline{n}}[h](\tau_y Z_{n_k})\Phi^0_{\underline{n}}[h](\tau_y Z_{n_k-m+1,2n_k-m})\right|M^{\otimes(2n_K-m)}dV_{2n_K-m}dX_{2,2n_K-m}\\ &\qquad\leq \frac{\|h\|^2}{n_k^m\mu_\varepsilon^{2n_k-m-1}}C^{n_k}\theta^{n_k-n_{k-1}}t^{m+n_{k-1}-1}.\end{aligned} \tag{4.3}$$

Indeed using Corollary 3.3 and the previous estimations,

$$\left|\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n_k})}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(t-k\theta)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right|$$
$$\leq \|g\|\|h\|\left(C^{n_k}\theta^{n_k-n_{k-1}}t^{n_{k-1}-1} + \left(\sum_{m=1}^{n_k}C^{n_k}\theta^{n_k-n_{k-1}}t^{n_{k-1}-1+m}\right)^{1/2}\right)$$
$$\leq \|g\|\|h\|C^{n_k}\theta^{(n_k-n_{k-1})/2}t^{n_k-1},$$

and in the same way,

$$\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n_K})}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_K}}(0)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon^c_\varepsilon}\right]=O\left(\varepsilon^{1/2}\|g\|\|h\|C^{n_k}t^{n_k-1}\right).$$

Summing on all possible $(n_1,\cdots,n_k)$,

$$\begin{aligned}|G^{\exp}_\varepsilon(t)|&\leq\sum_{k=1}^K\sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}}\sum_{n_k>2^k+n_{k-1}}\|g\|\|h\|C^{n_k}\theta^{(n_k-n_{k-1})/2}t^{n_k-1}\\ &\leq C\|g\|\|h\|\sum_{k=1}^K2^{k^2}\left(Ct\theta^{1/2}\right)^{2^k}\leq C\|g\|\|h\|\,t\,\theta^{1/2}\end{aligned}\tag{4.4}$$

because the series converges for θ small enough, and

$$\left|G^{\text{clust}}_\varepsilon(t)\right|\leq C\|g\|\|h\|\varepsilon^{1/2}+\sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}}\varepsilon^{1/2}\|g\|\|h\|C^{n_k}t^{n_k-1}\leq C\|g\|\|h\|\varepsilon^{1/2}2^{K^2}(Ct)^{2^K}.\tag{4.5}$$

This concludes the proof of (4.1).

Proof of (4.2). We recall that

$$\Phi^0_{\underline{n}}[h]=\Phi^{0,\theta}_{n_{K-1},n_K}\circ\Phi^{0,\theta}_{n_{K-2},n_{K-1}}\cdots\circ\Phi^{0,\theta}_{1,n_1}[h]=\frac{1}{n_k!}\sum_{(s_i,\bar{s}_i)_{i\leq n_k-1}}\prod_i\bar{s}_i\mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar{s}_i)}}h(\mathrm{Z}_{n_k}(k\theta))$$

and thus

$$\left|\Phi^0_{\underline{n}}[h]\right|\leq\frac{\|h\|}{n_k!}\sum_{(s_i,\bar{s}_i)_{i\leq n_k-1}}\mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar{s}_i)}}\tag{4.6}$$

where $\mathcal{R}^{\underline{n}}_{(s_i,\bar{s}_i)}\subset\mathcal{D}^{n_k}_\varepsilon$ is the set of initial parameters Z_{n_k} such that the pseudotrajectory $\mathrm{Z}_{n_k}(\tau,(s_i,\bar{s}_i)_i,(0)_j,Z_{n_k})$ has n_l remaining particles at time $(k-l)\theta$. Note that the left hand side of (4.6) is invariant under translations. Hence it is sufficient to fix $x_1=0$ and integrate with respect to (X_{2,n_k},V_{n_k}).

We define the *clustering tree* $T^>:=(\nu_i,\bar{\nu}_i)_{1\leq i\leq n_k-1}$ where the i-th collision happens between particles ν_i and $\bar{\nu}_i$ (and $\nu_i<\bar{\nu}_i$). Since in the present section pseudotrajectories have no recollision, the clustering tree is just the collision graph where we forget the collisions times (but not their order). It can be used to parameterize a partition of $\mathcal{R}^{\underline{n}}_{(s_i,\bar{s}_i)_i}$.

Let us fix a clustering tree. We perform the following change of variables

$$X_{2,n_k}\mapsto(\hat{x}_1,\cdots,\hat{x}_{n_k-1}),\ \forall i\in[1,n_k-1],\ \hat{x}_i:=x_{\nu_i}-x_{\bar{\nu}_i}$$

Fix then t_{i+1} the time of the $(i+1)$-th collision, as well as the relative positions $\hat{x}_1,\cdots,\hat{x}_{i-1}$. We denote $T_i=\theta$ if $i\leq n_k-n_{k-1}$, t else (at least n_k-n_{k-1} clustering collisions happen before time θ) and the i-th collision set as

$$B_{T^>,i}:=\left\{\hat{x}_i\middle|\exists\tau\in(0,T_i\wedge t_{i+1}),\ |\mathrm{x}_{\nu_i}(\tau)-\mathrm{x}_{\bar{\nu}_i}(\tau)|\leq\varepsilon\right\}.$$

Because particles $\mathrm{x}_{\nu_i}(\tau)$ and $\mathrm{x}_{\bar\nu_i}(\tau)$ are independent until their first meeting, we can perform the change of variable $\hat x_i \mapsto (t_i, \eta_i)$ where t_i is the first meeting time and

$$\eta_i := \frac{\mathrm{x}_{\nu_i}(t_i) - \mathrm{x}_{\bar\nu_i}(t_i)}{|\mathrm{x}_{\nu_i}(t_i) - \mathrm{x}_{\bar\nu_i}(t_i)|}.$$

This sends the Lebesgue measure $d\hat x_i$ to the measure $\mu_\varepsilon^{-1}((\mathrm{v}_{\nu_i}(t_i) - \mathrm{v}_{\bar\nu_i}(t_i))\cdot\eta_i)_+ d\eta_i dt_i$ and

$$\int \mathbb{1}_{B_{T^>,i}} d\hat x_i \le \frac{C}{\mu_\varepsilon} |\mathrm{v}_{\nu_i}(t_i) - \mathrm{v}_{\bar\nu_i}(t_i)| \int_0^{T_i \wedge t_{i+1}} dt_i.$$

We sum now on every possible edge $(\nu_i, \bar\nu_i)$:

$$\sum_{(\nu_i,\bar\nu_i)} |\mathrm{v}_{\nu_i}(t_i) - \mathrm{v}_{\bar\nu_i}(t_i)| \le 2n_k \sum_k |\mathrm{v}_k(t_i)| \le 2n_k \left(n_k \sum_k |\mathrm{v}_k(t_i)|^2 \right)^{1/2} \le n_k \left(n_k + \|V_{n_k}\|^2 \right)$$

using that the kinetic energy is decreasing for the pseudotrajectory. Hence

$$\begin{aligned}
\int \mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)}} d\hat x_1 \cdots d\hat x_{n_k-1} &\le \left(\frac{Cn_k}{\mu_\varepsilon}\right)^{n_k-1} \left(n_k + \|V_{n_k}\|^2\right)^{n_k-1} \int_0^{T_{n_k}} dt_{n_k} \cdots \int_0^{T_1\wedge t_2} dt_1 \\
&\le \left(\frac{Cn_k}{\mu_\varepsilon}\right)^{n_k-1} \left(n_k + \|V_{n_k}\|^2\right)^{n_k-1} \frac{t^{n_{k-1}-1}}{(n_{k-1}-1)!} \frac{\theta^{n_k-n_{k-1}}}{(n_k-n_{k-1})!} \\
&\le \left(\frac{6C}{\mu_\varepsilon}\right)^{n_k-1} \left(n_k + \|V_{n_k}\|^2\right)^{n_k-1} t^{n_{k-1}-1} \theta^{n_k-n_{k-1}},
\end{aligned}$$

using the Stirling's formula. For $A, B > 0$, $x \in \mathbb{R}$,

$$\left(A + x^2\right)^B e^{-\frac{x^2}{4}} = B^B \left(\frac{A+x^2}{B} e^{-\frac{A+x^2}{4B}}\right)^B e^{\frac{A}{4}} \le \left(\tfrac{4B}{e}\right)^B e^{\frac{A}{4}}.$$

Thus for some constant $C > 0$,

$$\int \left(n_k + \|V_{n_k}\|^2\right)^{n_k-1} e^{-\frac{\|V_{n_k}\|^2}{2}} dV_{n_k} \le (Cn_k)^{n_k-1} \int e^{-\frac{\|V_{n_k}\|^2}{4}} dV_{n_k} \le \left(2^{d/2} C n_k\right)^{n_k-1}$$

and

$$\begin{aligned}
\int \mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)}} M^{\otimes n_k} dX_{2,n_k} dV_{n_k} &\le \sum_{T^>} \int \prod_{i=1}^{n_k-1} \mathbb{1}_{B_{T^>,i}} d\hat x_i \; M^{\otimes n_k} dV_{n_k} \\
&\le C \left(\frac{C}{\mu_e}\right)^{n_k-1} t^{n_{k-1}-1} \theta^{n_k-n_{k-1}} \int \left(n_k + \|V_{n_k}\|^2\right)^{n_k-1} M^{\otimes n_k} dV_{n_k} \\
&\le C \left(\frac{C}{\mu_e}\right)^{n_k-1} t^{n_{k-1}-1} \theta^{n_k-n_{k-1}} n_k^{n_k-1},
\end{aligned}$$

(where the constants C change from line to line).

Finally we sum on the 4^{n_k-1} possible $(s_i, \bar s_i)_i$ and, dividing by the remaining $(n_k)!$, this gives the expected estimation. □

Proof of (4.3). We begin as in the previous paragraph

$$\Big|\Phi^0_{\underline{n}}[h](Z_{n_k})\Phi^0_{\underline{n}}[h](Z_{n_k-m+1,2n_k-m})\Big| \\ \leq \frac{\|h\|^2}{(n_k!)^2}\sum_{\substack{(s_i,\bar s_i)_{i\leq n_k-1}\\(s'_i,\bar s'_i)_{i\leq n_k-1}}} \mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)}}(Z_{n_k})\mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s'_i,\bar s'_i)}}(Z_{n_k-m+1,2n_k-m}).$$

We have to consider two pseudotrajectories $\mathrm{Z}(\tau) := \mathrm{Z}(\tau, Z_{n_k})$ and $\mathrm{Z}'(\tau) := \mathrm{Z}(\tau, Z_{n_k-m+1,2n_k-m})$. Note again that the right hand side is invariant under translation, hence we can fix $x_1 = 0$.

We construct the clustering tree $T^>$ as follows. We merge the collision graphs of the first and of the second pseudotrajectory. Then we look at edges one by one in temporal order, keeping only those which do not create a cycle. In this way we construct a tree which connects all the vertices. This leads to a graph with ordered edges. We finally remove then the non-clustering collisions and

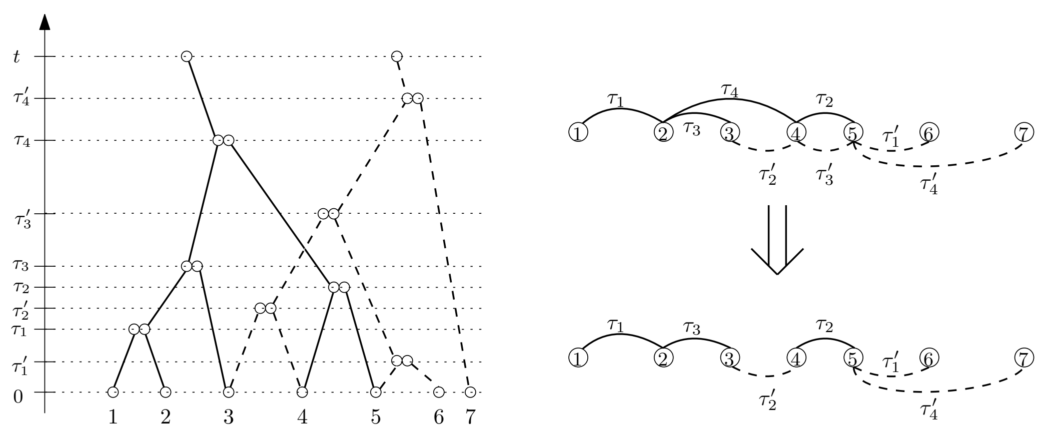

FIGURE 5. Example of construction of a clustering tree

obtain the clustering tree $T^> := (\nu_i, \bar\nu_i)$. As before, such trees induce a partition of

$$\Big\{Z_{2n_k-m} \in (\Lambda\times\mathbb{R}^d)^{2n_k-m} \,|\, Z_{n_k}\in\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)},\ (Z_{n_k-m+1,2n_k-m})\in\mathcal{R}^{\underline{n}}_{(s'_i,\bar s'_i)}\Big\}.$$

The rest of the proof is almost identical to the proof of (4.2). Fix the clustering tree, and perform the following change of variables

$$X_{2,2n_k-m}\mapsto(\hat x_1,\cdots,\hat x_{2n_k-m-1}),\ \forall i\in[1,2n_k-m-1],\ \hat x_i := x_{\nu_i}-x_{\bar\nu_i}.$$

Fix t_{i+1}, the time of the $(i+1)$-th collision and relative positions $\hat x_1,\cdots,\hat x_{i-1}$. We define the i-th collision sets as

$$B_{T^>,i} := \Big\{\hat x_i\Big|\exists\tau\in(0,T_i\wedge t_{i+1}),\ |\mathrm{x}_{\nu_i}(\tau)-\mathrm{x}_{\bar\nu_i}(\tau)|\leq\varepsilon\,\mathrm{or}\,|\mathrm{x}'_{\nu_i}(\tau)-\mathrm{x}'_{\bar\nu_i}(\tau)|\leq\varepsilon\Big\}$$

where $T_i = \theta$ for the $(n_k - n_{k-1})$ first collisions, t else. By the same computation as above,

$$\begin{aligned}\int \mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar{s}_i)}}(Z_{n_k})\mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s'_i,\bar{s}'_i)}}(Z_{n_k-m+1,2n_k-m})M^{\otimes(2n_k-m)}dX_{2,2n_k-m}dV_{2n_k-m}\\ \leq \sum_{T^>}\int\prod_{i=1}^{2n_k-m-1}\mathbb{1}_{B_{T^>,i}}d\hat{x}_i\ M^{\otimes(2n_k-m)}dV_{2n_k-m}\\ \leq C\left(\frac{C}{\mu_\varepsilon}\right)^{2n_k-m-1}t^{n_{k-1}+m-1}\theta^{n_k-n_{k-1}}(2n_k-m)^{2n_k-m-1}\\ \leq C\frac{(C)^{2n_k}}{\mu_\varepsilon^{2n_k-m-1}}t^{n_{k-1}+m-1}\theta^{n_k-n_{k-1}}n_k^{2n_k-m-1}.\end{aligned}$$

We sum on all the possible parameters $(s_i, \bar{s}_i)_i$ and $(s'_i, \bar{s}'_i)_i$ and get

$$\begin{aligned}\int\Big|\Phi^0_{\underline{n}}[h](Z_{n_k})\Phi^0_{\underline{n}}[h](Z_{n_k-m+1,2n_k-m})\Big|M^{\otimes(2n_k-m)}dX_{2,2n_k-m}dV_{2n_k-m}\\ \leq \|h\|^2\frac{n_k^{2n_k-m-1}C^{2n_k}}{(n_k!)^2\mu_\varepsilon^{2n_k-m-1}}t^{n_{k-1}+m-1}\theta^{n_k-n_{k-1}}\end{aligned}$$

which provides the announced result, by Stirling formula. □

5. Estimation of non-pathological recollisions

The objective of this section is to bound

$$G^{\mathrm{rec},1}_\varepsilon(t) := \sum_{\substack{1\leq k\leq K-1\\ 1\leq k'\leq K'}}\ \sum_{\substack{n_1\leq\cdots\leq n_k\\ n_j-n_{j-1}\leq 2^j}}\ \sum_{n''\geq n'\geq n_k}\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n''})}\Phi^{>,k'}_{\underline{n},n',n''}[h]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\ \zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right].$$

Proposition 5.1. *For ε small enough,*

$$\left|G^{rec,1}_\varepsilon(t)\right| \leq \|g\|\|h\|\varepsilon^{\alpha/2}(C't)^{2^{t/\theta}+2d+6}. \tag{5.1}$$

It is sufficient to prove the two following estimations:

Proposition 5.2. *Fix $k\in\mathbb{N}$, $\underline{n} := (n_1,\cdots,n_k)\in\mathbb{N}^k$ and $(n',n'')\in\mathbb{N}^2$ with $n_1\leq n_2\leq\cdots\leq n_k\leq n'\leq n''$. Then*

$$\int\sup_{y\in\Lambda}\left|\Phi^{>,k'}_{\underline{n},n',n''}[h](\tau_y Z_{n''})\right|M^{\otimes n''}dV_{n''}dX_{2,n''} \leq \varepsilon^\alpha\frac{\|h\|}{\mu_\varepsilon^{n''-1}}C^{n''}\theta^{(n''-n_k-2)_+}\delta^2 t^{n_k+2d+4}, \tag{5.2}$$

and, for $m\in[1,n'']$,

$$\begin{aligned}\int\sup_{y\in\Lambda}\left|\Phi^{>,k'}_{\underline{n},n',n''}[h](\tau_y Z_{n''})\Phi^{>,k'}_{\underline{n},n',n''}[h](\tau_y Z_{n''-m+1,2n''-m})\right|M^{\otimes(2n''-m)}dV_{2n''-m}dX_{2,2n''-m}\\ \leq \varepsilon^\alpha\frac{\|h\|^2}{(n'')^m\mu_\varepsilon^{2n''-m-1}}C^{n''}\theta^{(n''-n_k-2)_+}\delta^2 t^{n_k+2d+4+m}.\end{aligned} \tag{5.3}$$

Using these estimations and Corollary 3.3,

$$\begin{aligned}
&\left|\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n''})}\Phi^{>,k'}_{\underline{n},n',n''}[h]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right| \\
&\quad\leq \|h\|\|g\|\left(\varepsilon^\alpha C^{n''}\theta^{(n''-n_k-2)_+}\delta^2 t^{n_k+2d+4}+\left(\sum_{m=1}^{n''}\varepsilon^\alpha C^{n''}\theta^{(n''-n_k-2)_+}\delta^2 t^{n_k+2d+4+m}\right)^{1/2}\right) \\
&\quad\leq \|g\|\|h\|\delta\varepsilon^{\alpha/2}C^{n''}\theta^{(n''-n_k-2)_+/2}t^{\frac{n''+n_k}{2}+2d+4} \\
&\quad\leq \|g\|\|h\|\delta\varepsilon^{\alpha/2}(Ct)^{n_k+2d+5}(Ct\theta)^{(n''-n_k-2)_+/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|G^{\mathrm{rec},1}_\varepsilon(t)\right| &\leq \sum_{\substack{1\leq k\leq K-1\\1\leq k'\leq K'}}\ \sum_{\substack{n_1\leq\cdots\leq n_k\\n_j-n_{j-1}\leq 2^j}}\ \sum_{n''\geq n'\geq n_k}\|g\|\|h\|\delta\varepsilon^{\alpha/2}(Ct)^{n_k+2d+5}(Ct\theta)^{(n''-n_k-2)_+/2} \\
&\leq \|g\|\|h\|K'\delta\varepsilon^{\alpha/2}K^{K^2}(Ct)^{2^K+2d+5} \\
&\leq \|g\|\|h\|\varepsilon^{\alpha/2}(C't)^{2^K+2d+6}
\end{aligned}\tag{5.4}$$

using that $K'\delta\leq t$.

Proof of (5.2). We recall that the pseudotrajectory development takes the form

$$\begin{aligned}
\Phi^{>,k'}_{\underline{n},n',n''}[h](Z_{n''}) &= \Phi^{>,\delta}_{n',n''}\circ\Phi^{0,k'\delta}_{n_k,n'}\circ\Phi^0_{\underline{n}}[h](Z_{n''}) \\
&= \frac{1}{n''!}\sum_{\substack{((s_i,\bar s_i)_i,(\kappa_j)_j)\\\kappa_j\leq\gamma-1}}\prod_i \bar s_i\mathbb{1}_{\mathcal{R}_{((s_i,\bar s_i)_i,(\kappa_j)_j)}}h\big(\mathrm{Z}(k\theta+k'\delta,((s_i,\bar s_i)_i,(\kappa_j)_j),Z_{n''})\big)\ .
\end{aligned}$$

Here $\mathcal{R}^>_{((s_i,\bar s_i)_i,(\kappa_j)_j)}$ is the set of initial configurations $Z_{n''}$ such that the pseudotrajectory has:

- n' particles at time δ,
- n_l particles at time $k'\delta+(k-l)\theta$,
- at least one recollision,
- no recollision after time δ
- with no pathological recollision (thanks to the asymmetric conditioning).

Lemma 5.3. *There exist a constant $\alpha\in(0,1)$ such that for any $\underline{n}$, n', n'', k' and $((s_i,\bar s_i)_i,(\kappa_j)_j)$,*

$$\int\mathbb{1}_{\mathcal{R}_{((s_i,\bar s_i),(\kappa_j)_j)}}M^{\otimes n''}dX_{2,n''}dV_{n''}\leq C'\left(\frac{C'}{\mu_\varepsilon}\right)^{n''-1}(n'')^{n''}\delta^2\theta^{(n''-n_k-2)_+}t^{n_k+2d+4}\varepsilon^\alpha$$

Proof. We may define the clustering tree $T^>$ as before, by looking at collisions in temporal order and keeping only the clustering collisions. However, this will not be sufficient to characterize the initial data.

Let $(q,\bar q)$ (with $q<\bar q$) be the first two particles having a non-clustering collision, τ_{cycle} the time of this collision and $c\in[1,n''-1]$ such that τ_{cycle} lies between the times of the c-th and the $(c+1)$-th clustering collision. The parameters $(T^>,(q,\bar q,c))$ provide a partition of the set of initial data.

Considering the change of variables

$$\forall i\in[1,n''-1],\ \hat x_i:=x_{\nu_i}-x_{\bar\nu_i},\ X_{2,n''}\mapsto(\hat x_1,\cdots,\hat x_{n''-1})$$

with $T^> =: (\nu_i, \bar{\nu}_i)_{i \leq n''-1}$, we can construct as in the previous section a sequence of sets $B^i_{T^>,(q,\bar{q},c)}$ depending only on $V_{n''}$ and $\hat{x}_1, \cdots, \hat{x}_{i-1}$ which condition the relative position $\hat{x}_i$. The only difference is that the construction has to take into account the addition of one cycle. We define in the following $(T_i)_i$ by $T_i = \delta$ if i is smaller than $n'' - n'$, θ if i is between $n'' - n' + 1$ and $n'' - n_k$ and t else (T_i "counts" the number of clustering collisions in $[0, \delta]$, $[\delta, k'\delta]$ and $[k'\delta, k'\delta + k\theta]$).

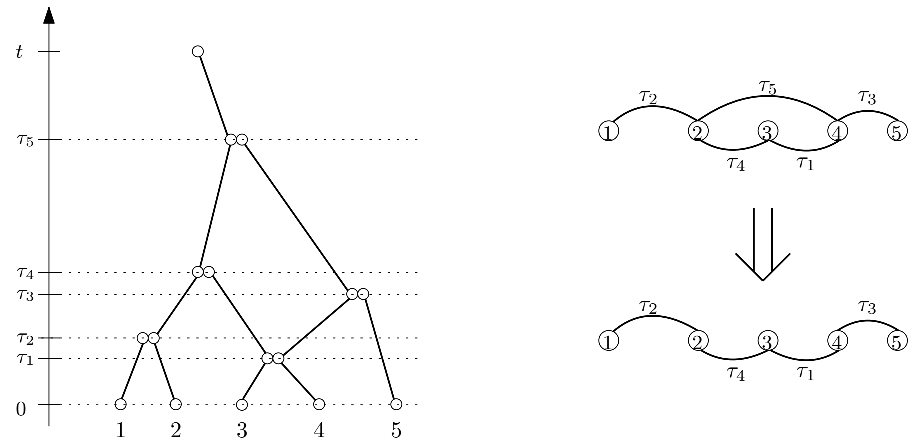

FIGURE 6. Example of construction of a clustering tree. Here $(q, \bar{q}, c) = (2, 4, 4)$.

We need to characterize one particular collision in $T^>$, which conditions the appearance of the non-clustering collision, and for which we have to adapt the definition of $B^i_{T^>,(q,\bar{q},c)}$.

Definition 5.1. We call *parent* p of a group of particles $(q_k)_k$ at time τ the p-th edge with the largest p such that one of the particles $(q_k)_k$ is deflected at $\tau_p \leq \tau$. If such a parent does not exist, then we set $\tau_p := 0$.

We define the *connector* k of two particles $(q, \bar{q})$ the index of the first edge, going backwards from τ to zero, realizing a connected path between q and $\bar{q}$.

The *tutor* j of two particles $(q, \bar{q})$ at time τ is the largest j with $t_j \leq \tau$ such that j is either the parent at time τ or the connector of $(q, \bar{q})$.

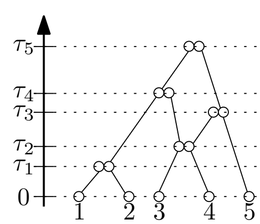

FIGURE 7. In this pseudotrajectory, the parent of particles $(2, 5)$ at time τ_5 is the collision between 1 and 2 at time τ_1 and the connector the collision between 3 and 4 at time τ_2.

Let j be the tutor of $(q,\bar q)$ before $\tau_{\rm cycle}$. We define

$$B^j_{T^>,(q,\bar q,c)} := \Big\{\hat x_j \Big| \exists \tau \in (0, T_j \wedge t_{j+1}),\ |{\rm x}_{\nu_j}(\tau) - {\rm x}_{\bar\nu_j}(\tau)| \leq \varepsilon \\ \text{and the } j\text{−th collision is the tutor of the cycle}\Big\}.$$

Note that, by construction, after the clustering time τ_j particles q and $\bar q$ do not change their velocities.

Proposition 5.4. *Assume that $d \geq 3$. Then, denoting by $w_q, w_{\bar q}, w_{q_j}, w_{\bar q_j}$ the velocities of $q, \bar q, q_j, \bar q_j$ at t^+_{j-1} (which are the same than at time t^-_j), and if the tutor j is the parent of $(q,\bar q)$, one has*

$$\int \mathbb{1}_{B^j_{T^>,(q,\bar q,c)}}\, d\hat x_j \leq \frac{C}{\mu_\varepsilon}(\mathbb{V}t)^d \times \left(\frac{\mathbb{V}\varepsilon|\log\varepsilon| \mathbb{1}_{q\neq \bar q_j}}{|w_q - w_{\bar q_j}|} + \frac{\mathbb{V}\varepsilon|\log\varepsilon| \mathbb{1}_{\bar q\neq \bar q_j}}{|w_{\bar q} - w_{\bar q_j}|} + \frac{\mathbb{V}t}{\mu_\varepsilon}\right)\ ; \tag{5.5}$$

otherwise if the tutor is a connector but not a parent,

$$\int \mathbb{1}_{B^j_{T^>,(q,\bar q,c)}}\, d\hat x_j \leq \frac{C}{\mu_\varepsilon}(\mathbb{V}\theta)^{d+1} \times \left[\sum_\zeta \mathbb{1}_{\sin(w_q - w_{\bar q},\zeta)\leq\varepsilon} + (\mathbb{V}\theta)^d \min\left(1, \frac{\varepsilon \mathbb{1}_{(q,\bar q)\neq(q_j,\bar q_j)}}{\sin\big(w_q - w_{\bar q}, w_{q_j} - w_{\bar q_j}\big)}\right)\right] \tag{5.6}$$

where the sum runs over $\zeta \in \mathbb{Z}^d \setminus \{0\}$ contained in the ball of radius $\mathbb{V}\theta$.

The above proposition uses the tutor to gain some smallness from the strong geometric constraint. However, the estimates in (5.5)-(5.6) lead to singularities in the relative velocities. Those singularities have to be integrated out either by using available parents (if any) or by using the Gaussian measure of the velocity distribution at time 0. The following proposition summarises the different possibilities.

Proposition 5.5. *(i) Let $q \neq \bar q$ be two particles of velocities $w_q, w_{\bar q}$ with parent ℓ. Let $\zeta \in \mathbb{Z}^d \setminus \{0\}$. Then one has that*

$$\int \left(\frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q - w_{\bar q_j}|} + \mathbb{1}_{\sin(w_q - w_{\bar q_j},\zeta)\leq\varepsilon}\right) \mathbb{1}_{B^\ell_{T^>,(q,\bar q,c)}}\, d\hat x_\ell \leq \frac{C}{\mu_\varepsilon}\, {}^{\mathbb{V}}\varepsilon|\log\varepsilon|\big(\delta \mathbb{1}_{\ell=1} + t\mathbb{1}_{\ell\neq1}\big)\,. \tag{5.7}$$

(ii) Let $q, \bar q, q_j, \bar q_j$ be particles with velocities $w_q, w_{\bar q}, w_{q_j}, w_{\bar q_j}$ and parent ℓ (say deflecting q), such that (q, q_j) and $(\bar q, \bar q_j)$ belong to different connected components of the dynamical graph.

$$\begin{aligned}\int \min\left(1, \frac{\varepsilon \mathbb{1}_{\{q,\bar q\}\neq\{q_j,\bar q_j\}}}{\sin\big(w_q - w_{\bar q}, w_{q_j} - w_{\bar q_j}\big)}\right) \mathbb{1}_{B^\ell_{T^>,(q,\bar q,c)}}\, d\hat x_\ell &\leq \frac{C}{\mu_\varepsilon}\,\mathbb{V}\varepsilon|\log\varepsilon|\,(\delta \mathbb{1}_{\ell=1} + t\mathbb{1}_{\ell\neq1}) \\ &\quad\times \left(1 + \frac{\theta\mathbb{V}\mathbb{1}_{(q,q_j)\text{ encounter at }\tau_\ell}}{|u_q + u_{q_j} - (w_{\bar q_j} + w_{\bar q})|} + \frac{t\mathbb{V}\mathbb{1}_{q=q_j}\mathbb{1}_{\bar q\neq\bar q_j}}{|w_{\bar q} - w_{\bar q_j}|}\right)\ ,\end{aligned} \tag{5.8}$$

denoting by u the pre-collisional velocities.
(iii) Let $q, \bar q, q_j, \bar q_j$ be particles with velocities $w_q, w_{\bar q}, w_{q_j}, w_{\bar q_j}$ such that $(q), (q_j)$ and $(\bar q, \bar q_j)$ belong to different connected components of the dynamical graph. Let ℓ be the first parent of $q, \bar q, q_j, \bar q_j$ deflecting only one particle of the group.

$$\int \frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q + w_{q_j} - (w_{\bar q_j} + w_{\bar q})|} \mathbb{1}_{B^\ell_{T^>,(q,\bar q,c)}}\, d\hat x_\ell \leq \frac{C}{\mu_\varepsilon}\,\mathbb{V}\varepsilon|\log\varepsilon|\big(\delta \mathbb{1}_{\ell=1} + t\mathbb{1}_{\ell\neq1}\big)\,. \tag{5.9}$$

(iv) For $q \neq \bar q$, $\zeta \in \mathbb{Z}^d \setminus \{0\}$

$$\begin{aligned}\int M(w_q)M(w_{q_j})M(w_{\bar q})M(w_{\bar q_j})\Big(\frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q-w_{\bar q}|}+\frac{\mathbb{V}\varepsilon|\log\varepsilon|}{|w_q+w_{q_j}-w_{\bar q}-w_{\bar q_j}|}+\mathbb{1}_{\sin(w_q-w_{\bar q},\zeta)\leq\varepsilon}\\ +\min\Big(1,\frac{\varepsilon\mathbb{1}_{(q,\bar q)\neq(q_j,\bar q_j)}}{\sin\big(w_q-w_{\bar q},w_{q_j}-w_{\bar q_j}\big)}\Big)\Big)dw_qdw_{q_j}dw_{\bar q}dw_{\bar q_j}\ \leq C\,\mathbb{V}\varepsilon|\log\varepsilon|\,.\end{aligned}\tag{5.10}$$

Propositions 5.4 and 5.5 have been proved in [7].

We can then integrate and sum on the $(\hat x_1,\cdots,\hat x_{n''-1})$ and $(T^>,(q,\bar q,c))$ and give a bound on

$$\sum_{(q,\bar q,c)}\sum_{T^>}\int d\hat x_1\mathbb{1}_{B^1_{T^>,(q,\bar q,c)}}\int d\hat x_2\cdots\int d\hat x_{n''-1}\mathbb{1}_{B^{n''-1}_{T^>,(q,\bar q,c)}}\,.$$

We integrate the constraints iteratively using successively Propositions 5.4 and 5.5: one obtains

$$\begin{aligned}&\int\mathbb{1}_{\mathcal{R}_{((s_i,\bar s_i),(\kappa_j)_j)}}M^{\otimes n''}dX_{2,n''}dV_{n''}\\ &\qquad\leq\left(\frac{C}{\mu_\varepsilon}\right)^{n''-1}(n'')^{2n''+2}\frac{\delta^{\max(n''-n',1)}}{\max(n''-n',1)!}\frac{\theta^{(n'-n_k-1)_+}}{(n'-n_k-1)_+!}\frac{t^{n_k}}{n_k!}(\mathbb{V}t)^{2d+4}\varepsilon|\log\varepsilon|\\ &\qquad\leq C'\left(\frac{C'}{\mu_\varepsilon}\right)^{n''-1}(n'')^{n''}\delta^2\theta^{(n''-n_k-2)_+}t^{n_k+2d+4}\varepsilon^{\alpha}\end{aligned}$$

using that $\mathbb{V}:=|\log\varepsilon|$. □

We obtain the expected result by summing on the

$$((s_i,\bar s_i)_i,(\kappa_j)_j)\in\{\pm1\}^{2n''-1}\times[0,\gamma-1]^{n''},$$

and dividing by $n''!$. □

Proof of (5.3). We use first the same bound of the previous section

$$\begin{aligned}&\left|\Phi^{>,k'}_{\underline{n},n',n''}[h](Z_{n''})\Phi^{>,k'}_{\underline{n},n',n''}[h](Z_{n''-m+1,2n''-m})\right|\\ &\leq\frac{\|h\|^2}{(n''!)^2}\sum_{\substack{((s_i,\bar s_i)_i,(\kappa_j)_j)\\ \kappa_j\leq\gamma-1}}\sum_{\substack{((s'_i,\bar s'_i)_i,(\kappa'_j)_j)\\ \kappa'_j\leq\gamma-1}}\mathbb{1}_{\mathcal{R}_{((s_i,\bar s_i)_i,(\kappa_j)_j)}}(Z_{n''})\mathbb{1}_{\mathcal{R}_{((s'_i,\bar s'_i)_i,(\kappa'_j)_j)}}(Z_{n''-m+1,2n''-m}).\end{aligned}$$

Note that the formula is invariant under translation. We can then fix $x_1=0$ and integrate with respect the other variables.

Fix $((s_i,\bar s_i)_i,(\kappa_j)_j)$ and $((s'_i,\bar s'_i)_i,(\kappa'_j)_j)$. There are two pseudotrajectories. We construct as in the proof of (5.2) the clustering tree $T_a^>$ and the recollision parameters $(q,\bar q,c)$ for the first recollision. We construct next the clustering graph $T_b^>$ of $\mathrm{Z}'(\tau)$ by induction. Let $(\nu_i,\bar\nu_i)_{i\leq I}$ be the edges of the collision graph of $\mathrm{Z}'(\tau)$, with temporal order. We begin by $T_0=\emptyset$. A the i-th step, we add $(\nu_i,\bar\nu_i)$ to T_{i-1} only if it does not create a cycle in the graph $T_a^>\cup T_{i-1}\cup\{(\nu_i,\bar\nu_i)\}$. At the end we have constructed the graph $T_b^>:=T_I$ and $T_a^>\cup T_b^>$ is a simply connected graph which connects all the particles. Note that $T_b^>$ has $n''-m$ edges.

We denote $T_a^>:=(\nu_i,\bar\nu_i)_{i\in[1,n''-1]}$ and $T_b^>:=(\nu_i,\bar\nu_i)_{i\in[n'',2n''-m-1]}$ (with $\nu_i<\bar\nu_i$) and we perform the change of variables

$$\forall i\in[1,2n''-m-1],\ \hat x_i:=x_{\nu_i}-x_{\bar\nu_i},\ X_{2,2n''-m}\mapsto(\hat x_1,\cdots,\hat x_{2n''-m-1}).$$

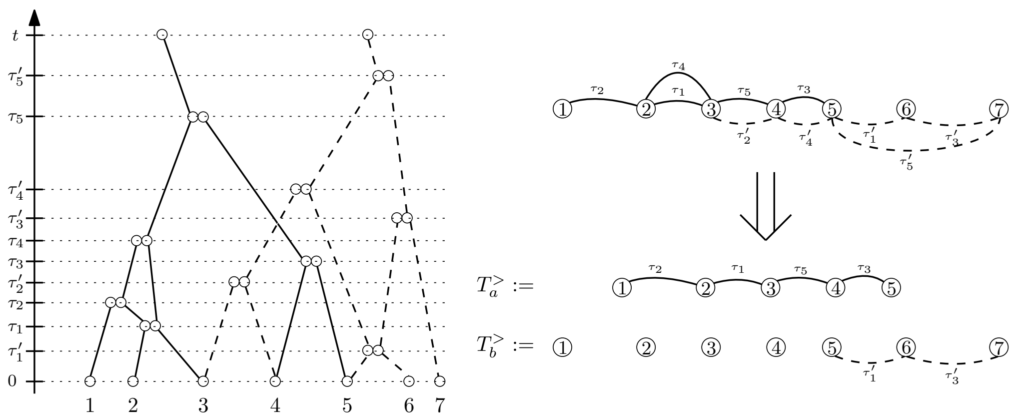

FIGURE 8. Example of construction of the clustering trees

Fixing $(\hat{x}_1, \cdots, \hat{x}_{n''-1})$, we construct first a sequence of clustering sets $B^i_{T_b^>}$ as in the proof of (4.3). Then we can reproduce the same strategy and

$$\sum_{T_b^>} \int_{B^{n''}_{T_b^>}} d\hat{x}_{n''} \cdots \int_{B^{2n''-m-1}_{T_b^>}} d\hat{x}_{2n''-m-1}$$
$$\leq \left(\frac{C'(2n''-m)}{\mu_\varepsilon} \right)^{n''-m} \left(\|V_{2n''-m}\|^2 + 2n'' - m \right)^{n''-m} \frac{t^{n''-m}}{(n''-m)!}.$$

Secondly, proceeding as in the proof of (5.2), we construct a sequence of clustering sets $B^i_{T_a^>,(q,\bar{q},c)}$ (for $i \leq n''-1$) of relative positions $\hat{x}_i$. Reproducing the same estimations,

$$\begin{aligned}
&\int \mathbb{1}_{\mathcal{R}_{((s_i,\bar{s}_i)_i,(\kappa_j)_j)}}(Z_{n''}) \mathbb{1}_{\mathcal{R}_{((s_i',\bar{s}_i')_i,(\kappa_j')_j)}}(Z_{n''-m+1,2n''-m}) M^{\otimes(n''-m)} dX_{2,2n''-m} dV_{2n''-m} \\
&\leq \sum_{\substack{(T_a^>,T_b^>)\\(q,\bar{q},c)}} \int M^{\otimes(n''-m)} dV_{2n''-m} \\
&\qquad \times \int_{B^1_{T^>,(q,\bar{q},c)}} d\hat{x}_1 \cdots \int_{B^{n''-1}_{T^>,(q,\bar{q},c)}} d\hat{x}_{n''-1} \int_{B^{n''}_{T_b^>}} d\hat{x}_{n''} \cdots \int_{B^{2n''-m-1}_{T_b^>}} d\hat{x}_{2n''-m-1} \\
&\leq \left(\frac{C}{\mu_\varepsilon} \right)^{2n''-m-1} (2n''-m)^{4n''-2m} \frac{\delta^{\max(n''-n',1)}}{\max(n''-n',1)!} \frac{\theta^{(n'-n_k-1)_+}}{(n'-n_k-1)_+!} \frac{t^{n_k+m}}{n_k!(n''-m)!} \\
&\qquad\qquad \times (\mathbb{V}t)^{2d+4} \varepsilon |\log \varepsilon| \\
&\leq C' \left(\frac{C'}{\mu_\varepsilon} \right)^{2n''-m-1} (2n''-m)^{2n''-m-1} \delta^2 \theta^{(n''-n_k-2)_+} t^{n_k+m+2d+4} \varepsilon^\alpha
\end{aligned}$$

where we use that for $(d_1, \cdots, d_k) \in \mathbb{N}^k$,

$$\frac{1}{d_1! \cdots d_k!} \leq \frac{k^{d_1+\cdots+d_k}}{(d_1+\cdots+d_k)!}$$

and the Stirling formula. Summing on the $(4\gamma)^{2(n''-1)}$ possible $((s_i,\bar{s}_i)_i,(\kappa_j)_j)$ and $((s_i',\bar{s}_i')_i,(\kappa_j')_j)$ and then dividing by $(n'')!^2$, we obtain the expected result. $\square$

6. Estimation of pathological recollisions

In the present section we discuss $G_\varepsilon^{\mathrm{rec},2}(t)$ defined by

$$\sum_{\substack{1\leq k\leq K-1\\ 1\leq k'\leq K'}}\ \sum_{\substack{n_1\leq\cdots\leq n_k\\ n_j-n_{j-1}\leq 2^j}}\Bigg(\sum_{n'\geq n_k}\mathbb{E}_\varepsilon\Bigg[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n'})}\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\Bigg]$$
$$-\sum_{n''\geq n'\geq n_k}\mathbb{E}_\varepsilon\Bigg[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n''})}\Phi^{\gamma,\delta}_{n'\leftarrow n''}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\Bigg]\Bigg).$$

We will prove the following bound:

Proposition 6.1. *For $\varepsilon>0$ small enough, we have*

$$\left|G_\varepsilon^{rec,2}(t)\right|\leq C\|h\|\|g\|\left(K2^{K^2}(Ct)^{2^{K+1}}\right)\varepsilon^{\alpha/2}. \tag{6.1}$$

6.1. Finite-parameter expansion.

In the sums

$$\sum_{(i_1,\cdots,i_{n'})}\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right.$$

and

$$\sum_{(i_1,\cdots,i_{n''})}\Phi^{\gamma}_{n'\leftarrow n''}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right),$$

the indicator function $\mathcal{X}_{\underline{i}_{n'}}(\mathbf{Z}_{\mathcal{N}})$ depends on all the particles of the system. In addition, in the first sum, the term $\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\right)$ depends on $\mathbf{Z}_{\mathcal{N}}(t_s)$ the position of all the particles at time t_s. In order to apply usual L^2 estimates, we first decompose these terms into a sum of functions evaluated on finitely many parameters: we want to construct two families of functions $(\Phi^{k'}_{\underline{n},n',p,l})_{p,l}$ and $(\Phi^{k'}_{\underline{n},n',n'',p})_p$ such that for almost $\mathbf{Z}_{\mathcal{N}}\in\mathcal{D}_\varepsilon$,

$$\sum_{(i_1,\cdots,i_{n'})}\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)=\sum_{n'\leq p\leq l}\sum_{(i_1,\cdots,i_l)}\Phi^{k'}_{\underline{n},n',p,l}(\mathbf{Z}_{\underline{i}_l}(t_s)),$$
$$\sum_{(i_1,\cdots,i_{n''})}\Phi^{\gamma}_{n'\leftarrow n''}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)=\sum_{n''\leq p}\sum_{(i_1,\cdots,i_p)}\Phi^{k'}_{\underline{n},n',n'',p}(\mathbf{Z}_{i_p}(t_s)).$$

6.1.1. *Decomposition of $\mathcal{X}_{(i_1,\cdots,i_{n'})}(\mathbf{Z}_{\mathcal{N}})$.*

We begin by expanding $\mathcal{X}_{(i_1,\cdots,i_{n'})}(\mathbf{Z}_{\mathcal{N}})$ as a sum of functions of a finite number of particles: we can decompose it formally as

$$\begin{aligned}\mathcal{X}_{(i_1,\cdots,i_n)}(\mathbf{Z}_{\mathcal{N}})&=1-\prod_{\substack{\varpi\subset\{1,\cdots,\mathcal{N}\}\\ \varpi\cap\{i_1,\cdots,i_n\}\neq\emptyset}}\left(1-\chi(\mathbf{Z}_\varpi)\right)\\&=-\sum_{p\geq n}\sum_{(i_{n+1},\cdots,i_p)}\frac{1}{(p-n)!}\sum_{\mathbf{p}\geq 0}\sum_{\bar\varpi\in\mathcal{Q}^{\mathbf{p}}_{[1,n],[n+1,p]}}\prod_{j=1}^{\mathbf{p}}\left[-\chi(\mathbf{Z}_{\underline{i}_{\varpi_j}})\right]\end{aligned}\tag{6.2}$$

where the function χ has been introduced in Definition 2.3, and we define, for ω_1 and ω_2 two subsets of $\mathbb{N}$ with empty intersection,

$$\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2} := \left\{ (\varpi_1, \cdots, \varpi_{\mathbf{p}}) \Big| \forall i,\ \varpi_i \subset \omega_1 \cup \omega_2,\ \varpi_i \cap \omega_1 \neq \emptyset\,;\ \omega_2 \subset \bigcup_{j=1}^{\mathbf{p}} \varpi_j\,;\ \forall i \neq j,\ \varpi_i \neq \varpi_j \right\}.$$

Defining

$$\mathfrak{X}_{n,p}\big(\mathbf{Z}_{\underline{i}_p}\big) := -\frac{1}{(p-n)!} \sum_{\mathbf{p}\geq 0} \sum_{\bar{\varpi} \in \mathcal{Q}^{\mathbf{p}}_{[1,n],[n+1,p]}} \prod_{j=1}^{\mathbf{p}} \Big[-\chi(\mathbf{Z}_{\underline{i}_{\varpi_j}}) \Big], \tag{6.3}$$

we have for any bounded and measurable function h_n and any times $\tau_1, \tau_2 \in \mathbb{R}$

$$\sum_{(i_1,\cdots,i_n)} h_n(\mathbf{Z}_{\underline{i}_n}(\tau_1)) \mathcal{X}_{\underline{i}_n}(\mathbf{Z}_{\mathcal{N}}) := \sum_{p\geq n} \sum_{(i_1,\cdots,i_p)} h_n\big(\mathbf{Z}_{\underline{i}_n}(\tau_1)\big) \mathfrak{X}_{n,p}\big(\mathbf{Z}_{\underline{i}_p}(\tau_2)\big).$$

Any family $(\varpi_1, \cdots, \varpi_{\mathbf{p}}) \in \mathcal{Q}^{\mathbf{p}}_{[1,n],[n+1,p]}$, has all its terms disjoint. Thus $\mathbf{p}$ is smaller than the cardinality of $\{\varpi,\ \varpi \subset \{1, \cdots, p\}\}$, 2^p. Thus $|\mathfrak{X}_{n,p}|$ is bounded by 2^{2^p}.

The preceding equality holds on $\{\mathcal{N} \leq N\}$ for every $N \in \mathbb{N}$ and the number of particles is bounded on Υ_ε. Hence the decomposition is valid on Υ_ε.

Applying the decomposition to the formula for $G^{\mathrm{rec},2}_\varepsilon(t)$ we obtain: for any $\mathbf{Z}_{\mathcal{N}} \in \mathcal{D}_\varepsilon$,

$$\begin{aligned} &\sum_{(i_1,\cdots,i_{n''})} \Phi^{\gamma}_{n'\leftarrow n''} \left[\Phi^{0,k'}_{\underline{n},n'}[h] \right] \big(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\big)\, \mathcal{X}_{\underline{i}_{n'}}\big(\mathbf{Z}_{\mathcal{N}}(t_s)\big) \\ &\qquad = \sum_{p\geq n''} \sum_{(i_1,\cdots,i_p)} \Phi^{\gamma}_{n'\leftarrow n''} \left[\Phi^{0,k'}_{\underline{n},n'}[h] \right] \big(\mathbf{Z}_{\underline{i}_{n''}}(t_s)\big)\, \mathfrak{X}_{n',p}(\mathbf{Z}_{\underline{i}_{[1,n']\cup[n''+1,p]}}(t_s)) \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} &\sum_{(i_1,\cdots,i_{n'})} \Phi^{0,k'}_{\underline{n},n'}[h] \big(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\big)\, \mathcal{X}_{\underline{i}_{n'}}\big(\mathbf{Z}_{\mathcal{N}}(t_s)\big) \\ &\qquad = \sum_{p\geq n'} \sum_{(i_1,\cdots,i_p)} \Phi^{0,k'}_{\underline{n},n'}[h] \big(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\big)\, \mathfrak{X}_{n',p}(\mathbf{Z}_{\underline{i}_p}(t_s)). \end{aligned} \tag{6.5}$$

6.1.2. *Dynamical cluster development.* In the second member of $G^{\mathrm{rec},2}_\varepsilon$ we look at functions at time t_s and $t_s - \delta$. To come back to one single evaluation time, we have to do some pseudotrajectory development. But tree pseudotrajectories are not handable, precisely because of possible many local recollision. Thus, we introduce a different kind of pseudotrajectory development, called here "dynamical cluster development" (see [23], from which we take inspiration).

We denote by $\mathsf{Z}^\lambda(\tau) = (\mathsf{X}^\lambda(\tau), \mathsf{V}^\lambda(\tau))$ the trajectory of the particles $\lambda \subset \mathbb{N}$ in a realization of the hard sphere dynamics associated to the Hamiltonian

$$\mathcal{H}_\lambda(Z_\lambda) := \sum_{q\in\lambda} \frac{|v_q|^2}{2} + \sum_{\substack{q,q'\in\lambda \\ q\neq q'}} \mathcal{V}\left(\frac{x_q - x_{q'}}{\varepsilon}\right)$$

-isolated of the other particles- with initial data Z_λ. For any subset $\lambda' \subset \lambda$, $\mathsf{Z}^\lambda_{\lambda'}(\tau)$ is the trajectory of particles λ' in $\mathsf{Z}^\lambda(\tau)$.

We say that $\mathsf{Z}^\lambda(\tau)$ forms a "dynamical cluster" if the collision graph on the time interval $[0,\delta]$ is connected, and we denote $\varphi_{|\lambda|}(Z_\lambda)$ the indicator function that the trajectory $\mathsf{Z}^\lambda(\tau)$ forms a cluster. In the same way, for $\lambda' \subset \lambda$, $\mathsf{Z}^\lambda(\tau)$ form a λ'-cluster if in the collision of $\mathsf{Z}^\lambda(\tau)$, all the particles are in the same connected components than one of the particles of λ'. The function $\varphi^{\lambda'}_{|\lambda|}(Z_\lambda)$ is equal to 1 if $\mathsf{Z}^\lambda(\tau)$ is a λ'-cluster, 0 else.

We say that trajectories $\mathsf{Z}^\lambda(\tau)$ and $\mathsf{Z}^{\lambda'}(\tau)$ (with $\lambda\cap\lambda'=\emptyset$) have an overlap if there exists a couple of particle $(i,i')\in\lambda\times\lambda'$ and some time $\tau\in[0,\delta]$, such that $|\mathsf{x}^\lambda_i(\tau)-\mathsf{x}^{\lambda'}_j(\tau)|\le\varepsilon$. Then we denote $\lambda \overset{\circ}{\sim} \lambda'$. We can decompose the initial data into dynamical clusters partition $(\lambda_1,\cdots,\lambda_{\mathbf{l}})$: each $\lambda_2,\cdots,\lambda_{\mathbf{l}}$ is a dynamical cluster, λ_1 is a $\underline{i}_m$-cluster and there is no interaction between particles of two distinct λ_i. For any $\mathbf{Z}_{\mathcal{N}}\in\mathcal{D}_\varepsilon$,

$$h_m(\mathbf{Z}_{\underline{i}_m}(\delta)) = \sum_{\mathbf{l}=1}^{\mathcal{N}} \sum_{\substack{\underline{i}_m\subset\lambda_1\\ (\lambda_2,\cdots,\lambda_{\mathbf{l}})\in\mathcal{P}^{\mathbf{l}-1}_{\lambda_1^c}}} h_m(\mathbf{Z}_{\underline{i}_m}(\delta))\varphi^{\underline{i}_m}_{\lambda_1}(\mathbf{Z}_{\lambda_1})\prod_{i=2}^{\mathbf{l}}\varphi_{|\lambda_i|}(\mathbf{Z}_{\lambda_i}) \underbrace{\prod_{1\le i<j\le\mathbf{l}}\left(1-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right)}_{\substack{\text{the indicator function that two}\\ \text{different clusters do not overlap}}}.$$

where we have denoted $\mathcal{P}^r_\omega$ the set of the unordered partitions $(\rho_1,\cdots,\rho_r)$ of the set ω.

For $(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathbf{l}}})\in\prod_{i=1}^{\mathbf{l}}\mathcal{D}^{|\lambda_i|}_\varepsilon$ initial data, we look at the indicator function that for any $i\neq j$, $\mathsf{Z}^{\lambda_i}(\tau)$ and $\mathsf{Z}^{\lambda_j}(\tau)$ have no overlap. As in Section 3, we can expand the function:

$$\prod_{1\le i<j\le\mathbf{l}}\left(1-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right) = \sum_{\substack{\omega\subset[1,l]\\ 1\in\omega}} \underbrace{\sum_{C\in\mathcal{C}(\omega)}\prod_{(i,j)\in E(C)} -\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}}_{:=\psi_{|\omega|}(Z_{\lambda_1},Z_{\lambda_{\omega(2)}},\cdots,Z_{\lambda_{\omega(|\omega|)}})} \prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}}\left(1-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right). \tag{6.6}$$

We have defined $(\psi_l)_l$ the cumulants of the overlap indicator. We make a partition of $\mathcal{D}_\varepsilon$ depending on the way particles interact on the time interval $[0,\delta]$: fixing $\mathcal{N}\in\mathbb{N}$ and $\underline{i}_m$,

$$\begin{aligned}&h_m(\mathbf{Z}_{\underline{i}_m}(\delta))\\ &=\sum_{\mathbf{l}=1}^{\mathcal{N}} \sum_{\substack{\underline{i}_m\subset\lambda_1\\ (\lambda_2,\cdots,\lambda_{\mathbf{l}})\in\mathcal{P}^{\mathbf{l}-1}_{\lambda_1^c}}} h_m(\mathbf{Z}_{\underline{i}_m}(\delta))\varphi^{\underline{i}_m}_{\lambda_1}(\mathbf{Z}_{\lambda_1})\prod_{i=2}^{\mathbf{l}}\varphi_{|\lambda_i|}(\mathbf{Z}_{\lambda_i})\sum_{\substack{\omega\subset[1,l]\\ 1\in\omega}}\psi_{|\omega|}(\mathbf{Z}_{\underline{\lambda}_\omega})\prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}}\left(1-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right).\end{aligned}$$

We make the change of variable

$$\left(\mathbf{l},(\lambda_1,\cdots,\lambda_{\mathbf{l}}),\omega\right)\mapsto\left(\rho,\mathbf{l}_1,\left(\bar\lambda_1,\cdots,\bar\lambda_{\mathbf{l}_1}\right),\mathbf{l}_2,\left(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathbf{l}_2}\right)\right)$$

where

$$\rho:=\bigcup_{i\in\omega}\lambda_i,\ \ \mathbf{l}_2:=|\omega|,\ \ \mathbf{l}_1:=\mathbf{l}-|\omega|,\ \ \left(\bar\lambda_1,\cdots,\bar\lambda_{\mathbf{l}_1}\right):=(\lambda_j)_{j\in\omega^c}\ \text{and}\ \left(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathbf{l}_2}\right):=(\lambda_j)_{j\in\omega}\,.$$

The set ρ is the set of particles which interact (in the dynamics or *via* an overlap) in $\underline{i}_m$. Thus

$$h_m(\mathbf{Z}_{\underline{i}_m}(\delta)) = \sum_{\rho\supset\underline{i}_m}\sum_{\mathfrak{l}_1=1}^{|\rho|}\sum_{\substack{\underline{i}_m\subset\bar\lambda_1\subset\rho\\(\bar\lambda_2,\cdots,\bar\lambda_{\mathfrak{l}_1})\in\mathcal{P}^{\mathfrak{l}_1-1}_{\bar\lambda_1^c}}} h_m\left(\mathsf{Z}^{\bar\lambda_1}_{\underline{i}_m}(\delta)\right)\varphi^{\underline{i}_m}_{\bar\lambda_1}\left(\mathbf{Z}_{\bar\lambda_1}\right)\prod_{i=2}^{\mathfrak{l}_1}\varphi_{|\bar\lambda_i|}\left(\mathbf{Z}_{\bar\lambda_i}\right)\psi_{\mathfrak{l}_1}\left(\mathbf{Z}_{\bar\lambda_1},\cdots,\mathbf{Z}_{\bar\lambda_{\mathfrak{l}_1}}\right)$$
$$\times\sum_{\mathfrak{l}_2=1}^{|\rho^c|}\sum_{(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathfrak{l}_2})\in\mathcal{P}^{\mathfrak{l}_2}_{\rho^c}}\prod_{i=1}^{\mathfrak{l}_2}\varphi_{|\tilde\lambda_i|}(\mathbf{Z}_{\tilde\lambda_i})\prod_{\substack{(i,j)\in(\omega^c)^2\\ i\neq j}}\left(1-\mathbb{1}_{\tilde\lambda_i\overset{\circ}{\sim}\tilde\lambda_j}\right).$$

The second line is the sum over all possible partitions $(\tilde\lambda_1,\cdots,\tilde\lambda_{\mathfrak{l}_2})$ of ρ^c of the indicator function that they are effectively the dynamical cluster of the initial data. Hence it is equal to one. Thus defining the n-th *dynamical cumulant* as

$$\mathsf{f}_{m\leftarrow n}[h_m](Z_n) := \frac{1}{(n-m)!}\sum_{\mathfrak{l}=1}^{n}\sum_{\substack{\lambda_1\subset[1,n]\\ [1,m]\subset\lambda_1}}\sum_{\substack{(\lambda_2,\cdots,\lambda_{\mathfrak{l}})\\ \in\mathcal{P}^{\mathfrak{l}-1}_{\lambda_1^c}}} h_m(\mathsf{Z}^{\lambda_1}_{[1,m]}(\delta))\psi_l(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathfrak{l}}})$$
$$\times\varphi^{[1,m]}_{|\lambda_1|}(Z_{\lambda_1})\prod_{i=2}^{\mathfrak{l}}\varphi_{|\lambda_i|}(Z_{\lambda_i}), \tag{6.7}$$

we obtain the dynamical cluster expansion:

Theorem 6.2. *For almost all $\mathbf{Z}_{\mathcal{N}}\in\mathcal{D}_\varepsilon$ we have*

$$h_m\left((\mathbf{Z}_{\underline{i}_m}(\delta)\right) = \sum_{n\geq m}\sum_{(i_{m+1},\cdots,i_n)}\mathsf{f}_{m\leftarrow n}[h_m]\left(\mathbf{Z}_{\underline{i}_n}(0)\right). \tag{6.8}$$

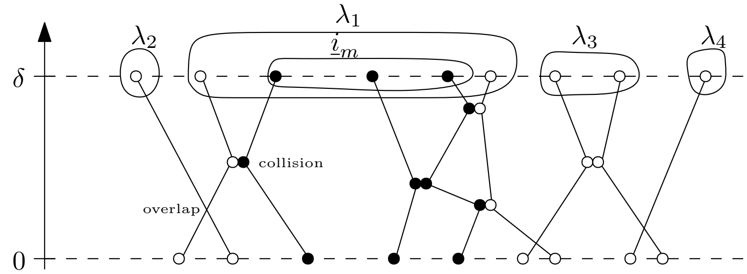

FIGURE 9. Example of trajectory in a dynamical cumulant. We want to follow the black particles.

Applying this to (6.4), for any $\mathbf{Z}_{\mathcal{N}}\in\Upsilon_\varepsilon$,

$$\sum_{(i_1,\cdots,i_{n'})}\Phi^{0,k'}_{\underline{n},n'}[h]\left(\mathbf{Z}_{\underline{i}_{n'}}(t_s+\delta)\right)\mathcal{X}_{\underline{i}_{n'}}\left(\mathbf{Z}_{\mathcal{N}}(t_s)\right)$$
$$=\sum_{p\geq n'}\sum_{(i_1,\cdots,i)}\Phi^{0,k'}_{\underline{n},n'}[h]\otimes\mathbb{1}^{\otimes p-n'}\left(\mathbf{Z}_{\underline{i}_p}(t_s+\delta)\right)\mathfrak{X}_{n',p}(\mathbf{Z}_{\underline{i}_p}(t_s))$$
$$=\sum_{l\geq p\geq n'}\sum_{(i_1,\cdots,i_l)}\mathsf{f}_{p\leftarrow l}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\otimes\mathbb{1}^{\otimes p-n'}\right]\left(\mathbf{Z}_{\underline{i}_l}(t_s)\right)\mathfrak{X}_{n',p}(\mathbf{Z}_{\underline{i}_p}(t_s)) \tag{6.9}$$

where $\Phi^{0,k'}_{\underline{n},n'}\otimes\mathbb{1}^{\otimes(p-n')}$ is the function $Z_p\mapsto\Phi^{0,k'}_{\underline{n},n'}(Z_{[1,n']})$.

Finally we symmetrize these two functions:

$$\Phi^{k'}_{\underline{n},n',p,l}(Z_l) := \frac{1}{l!}\sum_{\sigma\in\mathfrak{S}_l} \mathsf{f}_{p\leftarrow l}\Big[\Phi^{0,k'}_{\underline{n},n'}[h]\otimes \mathbb{1}^{\otimes p-n'}\Big]\big(Z_{\sigma([1,l])}\big)\,\mathfrak{X}_{n',p}(Z_{\sigma([1,p])}) \tag{6.10}$$

$$\Phi^{k'}_{\underline{n},n',n'',p}(Z_p) := \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p} \Phi^{\gamma}_{n'\leftarrow n''}\Big[\Phi^{0,k'}_{\underline{n},n'}[h]\Big]\big(Z_{\sigma([1,n''])}\big)\,\mathfrak{X}_{n',p}(Z_{\sigma([1,n']\cup[n''+1,p])})\,. \tag{6.11}$$

We have thus rewritten $G^{\mathrm{rec},2}_\varepsilon(t)$ as a function evaluated on finitely many variables:

$$\begin{aligned} G^{\mathrm{rec},2}_\varepsilon(t) = \sum_{\substack{1\le k\le K-1\\ 1\le k'\le K'}}\ \sum_{\substack{n_1\le\cdots\le n_k\\ n_j-n_{j-1}\le 2^j}}\ \sum_{n'\ge 0}\Bigg(&\sum_{\substack{l\ge 0\\ p\ge 0}}\mathbb{E}_\varepsilon\Bigg[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_l)}\Phi^{k'}_{\underline{n},n',p,l}(\mathbf{Z}_{\underline{i}_l}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\Bigg]\\ &-\sum_{\substack{n''\ge 0\\ p\ge 0}}\mathbb{E}_\varepsilon\Bigg[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_p)}\Phi^{k'}_{\underline{n},n',n'',p}(\mathbf{Z}_{\underline{i}_p}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\Bigg]\Bigg). \end{aligned} \tag{6.12}$$

6.2. **Geometrical estimation of local recollisions.** The aim of this part is to prove the following bound on $\Phi^{k'}_{\underline{n},n',n'',p}$ and $\Phi^{k'}_{\underline{n},n',p,l}$:

Proposition 6.3. *Fix $n_1\le\cdots\le n_k\le n'\le n''<p$. For $m\in\{1,\cdots p\}$ we have*

$$\int_{x_1=0}\sup_{y\in\Lambda}\big|\Phi^{k'}_{\underline{n},n',n'',p}(\tau_y Z_p)\big|M^{\otimes p}dX_{2,p}dV_p \le \frac{\|h\|}{\mu_\varepsilon^{p-1}}C^p\delta^2\varepsilon^\alpha\theta^{(p-n_k-2)_+}t^{n_k-1}, \tag{6.13}$$

$$\begin{aligned}\int_{x_1=0}\sup_{y\in\Lambda}\big|\Phi^{k'}_{\underline{n},n',n'',p}(\tau_y Z_p)\Phi^{k'}_{\underline{n},n',n'',p}(\tau_y Z_{p+1-m,2p-m})\big|M^{\otimes(2p-m)}dX_{2,2p-m}dV_{p-m}\\ \le \frac{\|h\|^2}{p^m\mu_\varepsilon^{2p-m-1}}C^p\delta^2\varepsilon^\alpha\theta^{(p-n_k-2)_+}t^{n_k-1+p-m}.\end{aligned} \tag{6.14}$$

In the same way if we fix $n_1\le\cdots\le n_k\le n'\le n''<p\le l$, for $m\in\{1,\cdots l\}$ we have

$$\int_{x_1=0}\sup_{y\in\Lambda}\big|\Phi^{k'}_{\underline{n},n',p,l}(\tau_y Z_l)\big|M^{\otimes l}dX_{l-1}dV_l \le \frac{\|h\|}{\mu_\varepsilon^{l-1}}C^l\delta^2\varepsilon^\alpha\theta^{(l-n_k-2)_+}t^{n_k-1}, \tag{6.15}$$

$$\begin{aligned}\int_{x_1=0}\sup_{y\in\Lambda}\big|\Phi^{k'}_{\underline{n},n',p,l}(\tau_y Z_p)\Phi^{k'}_{\underline{n},n',p,l}(\tau_y Z_{l+1-m,2l-m})\big|M^{\otimes(2l-m)}dX_{2,2l-m}dV_{2l-m}\\ \le \frac{\|h\|^2}{l^m\mu_\varepsilon^{2l-m-1}}C^l\delta^2\varepsilon^\alpha\theta^{(l-n_k-2)_+}t^{n_k-1+l-m}.\end{aligned} \tag{6.16}$$

From this and using the quasi-orthogonality estimates of Corollary 3.3, we obtain:

$$\begin{aligned}\left|\mathbb{E}_\varepsilon\Bigg[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_p}\Phi^{k'}_{\underline{n},n',n'',p}(\mathbf{Z}_{\underline{i}_p}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\Bigg]\right| &\le \|h\|\|g\|C^p\Big(\delta^2\varepsilon^\alpha\theta^{(p-n_k-2)_+}t^{n_k-1}\varepsilon^{\frac12}\\ &\qquad+\Big(p\delta^2\varepsilon^\alpha\theta^{(p-n_k-2)_+}t^{n_k-1+p}\Big)^{\frac12}\Big)\\ &\le \delta\varepsilon^{\alpha/2}\|h\|\,\|g\|C^p(\theta t)^{(p-n_k-2)_+/2}t^{n_k}\end{aligned} \tag{6.17}$$

and in the same way

$$\left|\mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{\underline{i}_l}\Phi^{k'}_{\underline{n},n',p,l}(\mathbf{Z}_{\underline{i}}(t_s))\,\zeta^0_\varepsilon(g)\mathbb{1}_{\Upsilon_\varepsilon}\right]\right|\leq\delta\varepsilon^{\alpha/2}\|h\|\|g\|C^l(\theta t)^{(l-n_k-2)_+/2}t^{n_k}. \tag{6.18}$$

Because θ tends to 0 as ε goes to zero, for ε small enough, the two previous series are summable with respect to respectively (l,n'',n') and (l,p,n'). We recall that $K'=\theta/\delta$ and we sum on k, $\underline{n}$ and k' to obtain that there exists a positive constant C depending only on the dimension and γ such that

$$\begin{aligned}\left|G^{\mathrm{rec},2}_\varepsilon(t)\right|&\leq C\delta\varepsilon^{\alpha/2}\|h\|\|g\|\frac{\theta}{\delta}\sum_{k=1}^K\sum_{\substack{n_1\leq\cdots\leq n_k\\ n_j-n_{j-1}\leq 2^j}}(Ct)^{n_k}\leq C\varepsilon^{\alpha/2}\|h\|\|g\|\sum_{k=1}^K 2^{k^2}(Ct)^{2^{k+1}}\\ &\leq C\|h\|\|g\|\left(K2^{K^2}(Ct)^{2^{K+1}}\right)\varepsilon^{\alpha/2}\end{aligned}\tag{6.19}$$

which concludes the proof of (6.1).

We have to prove four slightly different versions of the same inequality. We will do so in detail only for the first one, then we will explain how to adapt the others.

The proofs of (6.13), (6.14), (6.15) and (6.16) are very similar. We give only the full details for the proof of (6.13), and we only highlight the main differences for the other ones.

Proof of (6.13). We recall that

$$\Phi^{k'}_{\underline{n},n',n'',p}(Z_p):=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\Phi^\gamma_{n'\leftarrow n''}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(Z_{\sigma([1,n''])}\right)\mathfrak{X}_{n',p}(Z_{\sigma([1,n']\cup[n''+1,p])}).$$

In $\Phi^\gamma_{n'\leftarrow n''}\left[\Phi^{k'}_{\underline{n},n'}[h]\right](Z_{n''})\,\mathfrak{X}_{n',p}(Z_{[1,n']\cup[n''+1,p]})$ we see three sets of indices:

- $[1,n']$ the set of particles in "final" tree pseudotrajectories development,
- $[n'+1,n'']$ the particles added in the local tree development,
- $[n''+1,p]$ the particles which produce local recollisions.

Any permutation σ which sends $[1,n']$, $[n'+1,n'']$ and $[n''+1,p]$ onto themselves stabilizes $\Phi^\gamma_{n'\leftarrow n''}\left[\Phi^{k'}_{\underline{n},n'}[h]\right](Z_{n''})\,\mathfrak{X}_{n',p}(Z_{[1,n']\cup[n''+1,p]})$ and

$$\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)=\frac{n'!\,(n''-n')!\,(p-n'')!}{p!}\sum_{\substack{\underline{\omega}\in\mathcal{P}^3_p\\ |\omega_1|=n'\\ |\omega_2|=p-n''}}\Phi^\gamma_{n'\leftarrow n''}\left[\Phi^{k'}_{\underline{n},n'}\right](Z_{\omega_1},Z_{\omega_3})\,\mathfrak{X}_{n',p}(Z_{\omega_1},Z_{\omega_2}).$$

Let us develop $\Phi^\gamma_{n'\leftarrow n''}\left[\Phi^{k'}_{\underline{n},n'}[h]\right]$ and $\mathfrak{X}_{n',l}$. For $((s_i,\bar{s}_i),(\kappa_i))$ a set of recollision parameters, we denote $\mathcal{R}_{((s_i,\bar{s}_i),(\kappa_j))}\subset\mathcal{D}^{n''}_\varepsilon$ the set of initial data such that there is

- n' particles at time δ,
- n_k particles at time $(k'+1)\delta$ and
- n_j particles at time $(k'+1)\delta+(k-j)\theta$.

Then

$$\Phi^{k'}_{\underline{n},n',n'',p}(Z_p) = \frac{1}{p!}\sum_{((s_i,\bar{s}_i),(\kappa_j))}\sum_{\substack{\omega_1\sqcup\omega_2\sqcup\omega_3=[p]\\ |\omega_1|=n'\\ |\omega_2|=p-n''}}\sum_{\mathbf{p}\geq 1}\sum_{\underline{\varpi}\in\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}} -\mathbb{1}_{\mathcal{R}_{((s_i,\bar{s}_i),(\kappa_j))}}(Z_{\omega_1\cup\omega_3}) \times h\left(\mathrm{Z}_{\omega_1\cup\omega_3}(t-t_s)\right)\prod_{i=1}^{n''-1}\bar{s}_i\prod_{i=1}^{\mathbf{p}}(-\chi(Z_{\varpi_i})) \tag{6.20}$$

where

$$\mathrm{Z}_{\omega_1\cup\omega_3}(t-t_s) := \mathrm{Z}(t-t_s,((s_i,\bar{s}_i),(\kappa_j)),Z_{\omega_1\cup\omega_3})$$

and we have

$$\left|\Phi^{k'}_{\underline{n},n',n'',p}\right|(Z_p) \leq \frac{\|h\|}{p!}\sum_{((s_i,\bar{s}_i),(\kappa_j))}\sum_{\substack{\omega_1\sqcup\omega_2\sqcup\omega_3=[p]\\ |\omega_1|=n'\\ |\omega_2|=p-n''}}\sum_{\mathbf{p}\geq 1}\sum_{\underline{\varpi}\in\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}} \mathbb{1}_{\mathcal{R}_{((s_i,\bar{s}_i),(\kappa_j))}}(Z_{\omega_1\cup\omega_3}) \times\prod_{i=1}^{\mathbf{p}}\chi(Z_{\varpi_i}). \tag{6.21}$$

Figure 10 gives a representation of the trajectory of each particle.

Note that the right hand side is invariant under translation. Thus one can fix $x_1 = 0$ and integrate with respect to the other variable.

We need to count the set of collision parameters: $|\{(s_i,\bar{s}_i)_i\}| \leq 4^p$ and $|\{(\omega_1,\omega_2,\omega_3)\} \leq 3^p$. Using that the pseudotrajectory has no recollision after time δ, and that particle can only meet $\gamma-1$ particles before δ, one can bound $\{(\kappa_j)_j\}| \leq (\gamma-1)^p$. The problem is the set of parameters $\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}$, which is huge: $|\bigcup_{\mathbf{p}}\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}| \lesssim 2^{2^p}$. We need now the global conditioning in order to control the number of acceptable $\underline{\varpi}$.

Given Z_p, we introduce $\underline{\rho} := (\rho_1,\cdots,\rho_{\mathbf{r}})$ the $\gamma\delta\mathbb{V}$-distance partition: consider the graph G with vertices $[1,p]$ with $(i,j)\in E(G)$ if and only if $|x_i - x_j| \leq \gamma\delta\mathbb{V}$ (we recall that $\gamma\in\mathbb{N}$ is a constant fixed in Section 2.3)). The ρ_i are the connected components of G. We define $\mathcal{D}^{\underline{\rho}}_{\varepsilon}\subset\mathcal{D}^p_{\varepsilon}$ as the set such that $\underline{\rho}$ is the distance partition, and the $\left(\mathcal{D}^{\underline{\rho}}_{\varepsilon}\right)_{\underline{\rho}}$ form a partition of $\mathcal{D}^p_{\varepsilon}$.

Inside each cluster ρ_i, a particle can only interact with the other particles of ρ_i as the kinetic energy $\|V_{\rho_i}(\tau)\|^2$ is bounded by $\mathbb{V}^2$. Hence the system ρ^i is isolated on $[0,\delta]$ and for any $\varpi\subset\omega_1\cup\omega_3$, if particles in Z_ω can have a pseudotrajectory with connected collision graph (and a local recollision), then there exists some ρ_i containing ω.

We can perform now the following parametrisation: for any ρ_i, we consider

- $\underline{\omega}^i := (\omega^i_1,\omega^i_2,\omega^i_3)$ the partition of ρ_i defined by $\omega^i_j := \omega_j\cap\rho_i$,
- $\underline{\varpi}^i := \{\varpi_j \text{ such that } \varpi_j\subset\rho_i\}$,
- $\mathfrak{p}_i := (\underline{\omega}^i,\underline{\varpi}^i)$, and $\mathfrak{P}(\rho_i)$ the set of possible $\mathfrak{p}_i$.

Because ρ_i is of size at most γ, there exists a constant C_γ depending only on γ such that $|\mathfrak{P}(\rho_i)| \leq C_\gamma$. Hence, the set of parameter $\prod_i\mathfrak{P}_i$ is of size smaller than $C^{\mathbf{r}}_\gamma \leq C^p_\gamma$.

Any particle in ω_2 or ω_3 has to be close to a particle in ω_1 because they are involved in some pseudotrajectory on $[0,\delta]$ involving a particle in ω_1. So for any ρ_i, ω^i_1 is not empty. Finally note that if we fix $\underline{\rho}$, the map $(\underline{\omega},\underline{\varpi})\mapsto(\mathfrak{p}_i)_i$ is onto.

We have now the following bound

$$\left|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\right| \leq \frac{\|h\|}{p!}\sum_{\mathbf{r}=1}^{p}\sum_{\underline{\rho}\in\mathcal{P}^{\mathbf{r}}_p}\sum_{\substack{((s_i,\bar{s}_i),(\kappa_j))\\ \underline{\mathfrak{p}}\in\prod_i\mathfrak{P}(\rho_i)}}\mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i,\bar{s}_i),(\kappa_i)}}(Z_p)\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})$$

where the function

$$\Delta_{\mathfrak{p}_i}(Z_{\rho_i}) := \mathbb{1}_{\substack{Z_{\rho_i}\text{ form a}\\ \text{distance cluster}}} \prod_{j=1}^{|\varpi^i|} \chi(Z_{\varpi^i_j})$$

controls the local cluster, and the relative position between clusters is controlled by

$$\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i,\bar{s}_i),(\kappa_i))} := \Big\{ Z_p \in \mathcal{D}^{\underline{\rho}}_{\varepsilon},\ Z_{\omega_1\cup\omega_3} \in \mathcal{R}_{((s_i,\bar{s}_i),(\kappa_i)} \Big\}.$$

The main idea is to freeze the relative position of each particle inside each ρ_i, and perform the change of variable

$$X_{2,p} \to ((x_{\min \rho_i})_{2\le i\le \mathbf{r}}, (\bar{X}_i)_{i\le \mathbf{r}}),\ \ \bar{X}_i := (x_j - x_{\min\rho_i})_{j\in\rho_i}.$$

Then we use the condition $\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i,\bar{s}_i),(\kappa_i)}$ to integrate the $(x_{\min\rho_i})_i$, using the same method than in [6] to control the first condition. The sum on $(\bar{X}_i)_i$ will be controlled later, using the condition $\Delta_{\mathfrak{p}_i}(Z_{\rho_i})$.

Lemma 6.4. *Fixing $\underline{\mathfrak{p}}$, $\underline{\rho}$, $((s_i,\bar{s}_i)_i,\kappa_j)$ and the coordinates V_p and $(\bar{X}_i)_{i\le\mathbf{r}}$, one obtains the following bound:*

$$\int \mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i,\bar{s}_i),(\kappa_i)}} e^{-\frac{1}{4}\|V_p\|^2} dx_{\min\rho_2}\cdots d\hat{x}_{\min\rho_\mathbf{r}} \le \frac{C^p t^{n_k\wedge \mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}}. \tag{6.22}$$

Proof. We define now the *clustering tree* of the distance cluster: For a pseudotrajectory $\mathrm{Z}_{\omega_1\cup\omega_3}(\tau)$, consider its collision graph $\mathcal{G}^{[0,t-t_s]}_{\omega_1\cup\omega_3}$. Then, the graph G is constructed by identifying in $\mathcal{G}^{[0,t-t_s]}_{\omega_1\cup\omega_3}$ the particles in a same cluster ρ_i. Finally, we define the clustering trees $T^> := (\nu_i,\bar{\nu}_i)_{1\le i\le \mathbf{r}-1}$ where the i-th clustering collision in G happens between cluster ρ_{ν_i} and $\rho_{\bar{\nu}_i}$.

We construct now a different representation of collision graphs. Let L_0 be equal to $\{\{1\},\cdots,\{\mathbf{r}\}\}$ and construct the L_i and $(\nu_{(i)},\bar{\nu}_{(i)})$ sequentially. Suppose that $L_{i-1} = (c_1,\cdots,c_l)$, the (c_j) forming a partition of $[1,r]$. The i-th collision happens between cluster $\nu_i\in c_a$ and $\bar{\nu}_i\in c_b$. Then:

- $L_i := \big(L_{i-1}\setminus\{c_a,c_b\}\big)\cup\{c_a\cup c_b\}$,
- $\{\nu_{(i)},\bar{\nu}_{(i)}\} := \{c_a,c_b\}$ with $\max\nu_{(i)} < \max\bar{\nu}_{(i)}$.

The $(\nu_{(i)},\bar{\nu}_{(i)})$ define a partition of $\mathcal{T}^>_\mathbf{r}$ (the set of ordered trees on $[1,\mathbf{r}]$).

We apply then the following change of variables:

$$\forall i\in\{1,\cdots,\mathbf{r}-1\},\ \hat{x}_i := x_{\min\nu_{(i)}} - x_{\min\bar{\nu}_{(i)}},$$
$$(x_{\min\rho_2},\cdots x_{\min\rho_\mathbf{r}}) \mapsto (\hat{x}_1\cdots,\hat{x}_{\mathbf{r}-1}).$$

We need to count the number of clustering collisions of $T^>$ happening between time δ and time θ. If $\mathbf{r} > n_k$, all the $\mathbf{r}-1$ collisions in $T^>$ cannot correspond to the n_k-1 annihilations of the time interval $[(k'+1)\delta, t-t_s]$. Thus at least $(\mathbf{r}-n_k)_+$ collision happen in $[\delta,(k'+1)\delta]\subset[0,2\theta]$.

Fix t_{i+1} the time of the $(i+1)$-th clustering collision and the relative positions $\hat{x}_{i-1},\cdots,\hat{x}_1$. We define the i-th clustering set

$$B_i := \bigcup_{\substack{q\in\bigcup_{j\in\nu_{(i)}}\rho_j\\ \bar{q}\in\bigcup_{\bar{j}\in\bar{\nu}_{(i)}}\rho_{\bar{j}}}} B_i^{q,\bar{q}}$$

with

$$B_i^{q,\bar{q}} := \Big\{\hat{x}_i\ \Big|\ \exists t_i\in[0,t_{i+1}\wedge T_i],\ |\mathrm{x}_{\bar{q}}(t_i) - \mathrm{x}_{\bar{q}}(t_i)| = \varepsilon\Big\}$$

and $T_i := 2\theta$ for the $(\mathbf{r}-n_k)_+$ first collisions, t else.

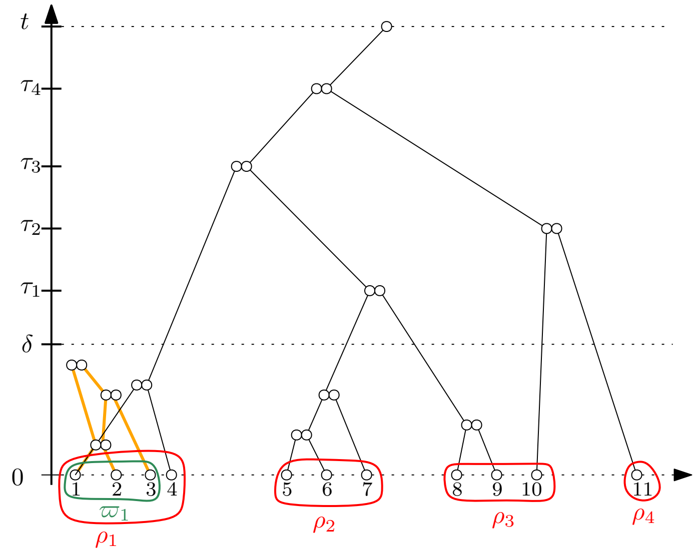

FIGURE 10. Example of a configuration of particles, its pseudotrajectory and the construction of its clustering tree. In this example, we set $\omega_1 = \{4, 6, 8, 10, 11\}$, $\omega_2 = \{1, 5, 7, 9\}$, and $\omega_3 = \{2, 3\}$. The black pseudotrajectory (involving particle $\omega_1 \cap \omega_2$) gives the position of the last existing particle 6. For the particle in $\varpi_1 = \{1, 2, 3\}$ one can find pseudotrajectory parameters such that the associated trajectory (represented in orange) has one recollision. We construct the distance cluster $(\rho_i)_i$, and its clustering tree $\mathcal{T}_> = ((2,3),(3,4),(1,2))$. The collision at time τ_4 is not clustering.

Up to time t_i the curve x_q and $\mathrm{x}_{\bar{q}}$ are independent. Hence we can perform the change of variable $\hat{x}_i \mapsto (t_i, \eta_i)$ with t_i the minimal collision time and

$$\eta_i = \frac{\mathrm{x}_{\bar{q}}(t_i) - \mathrm{x}_{\bar{q}}(t_i)}{|\mathrm{x}_{\bar{q}}(t_i) - \mathrm{x}_{\bar{q}}(t_i)|}.$$

The Jacobian of this diffeomorphism is $\mu_\varepsilon^{-1}|(\mathrm{v}_{\bar{q}}(t_i) - v_{\bar{q}}(t_i)) \cdot \eta_i| dt_i d\eta_i$. We integrate and we apply Cauchy-Schwartz inequality, using that the kinetic energy associated with cluster $\rho_{\nu_{(i)}}$ is non-increasing (we can only remove particles) up to time t_i.

Note that

$$\begin{aligned}\sum_{\substack{q\in\nu_{(i)}\\ \bar{q}\in\bar{\nu}_{(i)}}} \|\mathrm{v}_{\bar{q}}(t_i) - \mathrm{v}_{\bar{q}}(t_i)\| &\leq \|\mathrm{V}_{\rho_{\nu_{(i)}}}(t_i)\| \, |\rho_{\nu_{(i)}}|^{1/2} |\rho_{\bar{\nu}_{(i)}}| + \|\mathrm{V}_{\rho_{\bar{\nu}_{(i)}}}(t_i)\| \, |\rho_{\bar{\nu}_{(i)}}|^{1/2} |\rho_{\bar{\nu}_{(i)}}| \\ &\leq \Big(|\rho_{\nu_{(i)}}| + \|V_{\rho_{\nu_{(i)}}}\|^2\Big)\Big(|\rho_{\bar{\nu}_{(i)}}| + \|V_{\rho_{\bar{\nu}_{(i)}}}\|^2\Big) \\ &\leq \sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar{\nu}_i\in\bar{\nu}_{(i)}}} \Big(|\rho_{\nu_i}| + \|V_{\rho_{\nu_i}}\|^2\Big)\Big(|\rho_{\bar{\nu}_i}| + \|V_{\rho_{\bar{\nu}_i}}\|^2\Big).\end{aligned}$$

This gives the following bound on $|B_i|$

$$\begin{aligned}|B_i| &\leq \frac{C}{\mu_\varepsilon}\int_0^{t_{i+1}\wedge T_i} dt_i \sum_{q,\bar q} \|\mathrm{v}_{\bar q}(t_i)-\mathrm{v}_{\bar q}(t_i)\|\\ &\leq \frac{C}{\mu_\varepsilon}\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar\nu_i\in\bar\nu_{(i)}}}\Big(|\rho_{\nu_i}|+\|V_{\rho_{\nu_i}}\|^2\Big)\Big(|\rho_{\bar\nu_i}|+\|V_{\rho_{\bar\nu_i}}\|^2\Big)\int_0^{t_{i+1}\wedge T_i} dt_i.\end{aligned}$$

Permuting the product and the sum,

$$\begin{aligned}&\sum_{(\nu_{(i)},\bar\nu_{(i)})}\prod_{i=1}^{\mathbf{r}-1}\Big(|\rho_{\nu_{(i)}}|+\|V_{\rho_{\nu_{(i)}}}\|^2\Big)\Big(|\rho_{\bar\nu_{(i)}}|+\|V_{\rho_{\bar\nu_{(i)}}}\|^2\Big)\\ &\qquad\qquad = \sum_{(\nu_{(i)},\bar\nu_{(i)})}\prod_{i=1}^{\mathbf{r}-1}\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar\nu_i\in\bar\nu_{(i)}}}\Big(|\rho_{\nu_i}|+\|V_{\rho_{\nu_i}}\|^2\big)\big(|\rho_{\bar\nu_i}|+\|V_{\rho_{\bar\nu_i}}\|^2\Big)\\ &\qquad\qquad = \sum_{(\nu_i,\bar\nu_i)}\prod_{i=1}^{\mathbf{r}-1}\big(|\rho_{\nu_i}|+\|V_{\rho_{\nu_i}}\|^2\big)\big(|\rho_{\bar\nu_i}|+\|V_{\rho_{\bar\nu_i}}\|^2\big).\end{aligned}$$

Using that

$$\forall a,b\in\mathbb{N},\ \frac{(a+b)!}{a!b!}\leq 2^{a+b},$$

we have

$$\int_0^t dt_{\mathbf{r}-1}\cdots\int_0^{t_2\wedge T_2} dt_1 \leq \frac{t^{n_k\wedge\mathbf{r}-1}}{(n_k\wedge\mathbf{r}-1)!}\frac{\theta^{(\mathbf{r}-n_k)_+}}{((\mathbf{r}-n_k)_+)!}\leq 2^{\mathbf{r}-1}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mathbf{r}-1)!}.$$

We can sum now on the clustering collisions:

$$\begin{aligned}\int \mathbb{1}_{\mathcal{R}^{\rho,\mathfrak{p}}_{((s_i,\bar s_i),(\kappa_i)}} d\hat x_1\cdots\hat x_{\mathbf{r}-1} &\leq \sum_{(\nu_{(i)},\bar\nu_{(i)})}\int d\hat x_1'\mathbb{1}_{B_1}\int d\hat x_2'\cdots\int d\hat x_{\mathbf{r}-1}\mathbb{1}_{B_{\mathbf{r}-1}}\\ &\leq\left(\frac{C}{\mu_\varepsilon}\right)^{\mathbf{r}-1}\int_0^t dt_{\mathbf{r}-1}\cdots\int_0^{t_2\wedge T_2} dt_1\sum_{(\nu_{(i)},\bar\nu_{(i)})}\prod_{i=1}^{\mathbf{r}-1}\Big(|\rho_{\nu_{(i)}}|+\|V_{\rho_{\nu_{(i)}}}\|^2\Big)\Big(|\rho_{\bar\nu_{(i)}}|+\|V_{\rho_{\bar\nu_{(i)}}}\|^2\Big)\\ &\leq\left(\frac{2C}{\mu_\varepsilon}\right)^{\mathbf{r}-1}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mathbf{r}-1)!}\sum_{(\nu_i,\bar\nu_i)}\prod_{i=1}^{\mathbf{r}-1}\Big(|\rho_{\nu_i}|+\|V_{\rho_{\nu_i}}\|^2\Big)\Big(|\rho_{\bar\nu_i}|+\|V_{\rho_{\bar\nu_i}}\|^2\Big).\end{aligned}$$

Then denoting $d_i(G)$ the degree of vertices in a graph, $\mathcal{T}_\mathbf{r}$ the set of minimally (not oriented) connected graphs on $[1,\mathbf{r}]$,

$$\begin{aligned}\int \mathbb{1}_{\mathcal{R}^{\rho,\mathfrak{p}}_{((s_i,\bar s_i),(\kappa_i)}} d\hat x_1\cdots\hat x_{\mathbf{r}-1} &\leq\left(\frac{2C}{\mu_\varepsilon}\right)^{\mathbf{r}-1}\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{(\mathbf{r}-1)!}\sum_{T^>\in\mathcal{T}_\mathbf{r}^>}\prod_{i=1}^{\mathbf{r}}\Big(|\rho_i|+\|V_{\rho_i}\|^2\Big)^{d_i(T^>)}\\ &\leq\left(\frac{2C}{\mu_\varepsilon}\right)^{\mathbf{r}-1} t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}\sum_{T\in\mathcal{T}_\mathbf{r}}\prod_{i=1}^{r}\Big(|\rho_i|+\|V_{\rho_i}\|^2\Big)^{d_i(T)}.\end{aligned}$$

For $A,B>0$, $x\in\mathbb{R}$, there exists a constant $C>0$ such that

$$\big(A+x^2\big)^B e^{-\frac{x^2}{4}}\leq\big(\tfrac{4B}{e}\big)^B e^{\frac{A}{4}}.$$

We use this inequality to bound

$$\begin{aligned}\int \mathbb{1}_{\mathcal{R}^{\rho,\mathfrak{p}}_{((s_i,\bar s_i),(\kappa_i)}} & e^{-\frac14\|V_p\|^2}d\hat x_1\cdots\hat x_{\mathbf{r}-1}\\ &\leq \left(\frac{C}{\mu_\varepsilon}\right)^{\mathbf{r}-1} t^{n_k\wedge r-1}\theta^{(\mathbf{r}-n_k)_+}\sum_{T\in\mathcal{T}_\mathbf{r}}\prod_{i=1}^{\mathbf{r}}\Big(|\rho_i|+\|V_{\rho_i}\|^2\Big)^{d_i(T)}e^{-\frac14\sum_{i=1}^{\mathbf{r}}\|V_{\rho_i}\|^2}\\ &\leq \tilde C^l\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}}\sum_{T\in\mathcal{T}_\mathbf{r}}\prod_{i=1}^{\mathbf{r}}d_i(T)^{d_i(T)}\,.\end{aligned}\tag{6.23}$$

Next we use that for fixed $(d_1,\cdots,d_\mathbf{r})$ such that $\sum_i d_i=2(n-1)$,

$$\big|\{T\in\mathcal{T}_\mathbf{r}\big|\forall i\leq\mathbf{r},\,d_i(T)=d_i\}\big|=\frac{(\mathbf{r}-2)!}{(d_1-1)!\cdots(d_\mathbf{r}-1)!}\tag{6.24}$$

(see section 2 of [6]), which leads to

$$\begin{aligned}\sum_{T\in\mathcal{T}_\mathbf{r}}\prod_{i=1}^{\mathbf{r}}d_i(T)^{d_i(T)}&=(\mathbf{r}-2)!\sum_{\substack{d_1,\cdots,d_\mathbf{r}\\ \mathbf{r}-1\geq d_i\geq1\\ \sum_i d_i=2(\mathbf{r}-1)}}\prod_{i=1}^{\mathbf{r}}\frac{d_i^{d_i}}{(d_i-1)!}\\ &\leq(\mathbf{r}-2)!C^\mathbf{r}\sum_{\substack{d_1,\cdots,d_{\mathbf{r}-1}\\ \mathbf{r}-1\geq d_i\geq1\\ r-1\leq\sum_i d_i\leq2\mathbf{r}-3}}1\\ &\leq C^\mathbf{r}(\mathbf{r}-2)!\frac{(2\mathbf{r}-3)^{\mathbf{r}-1}}{(\mathbf{r}-1)!}\leq\tilde C^p(\mathbf{r}-1)!.\end{aligned}\tag{6.25}$$

Finally,

$$\int\mathbb{1}_{\mathcal{R}^{\rho,\mathfrak{p}}_{((s_i,\bar s_i),(\kappa_i)}}e^{-\frac14\|V_p\|^2}d\hat x_1\cdots\hat x_{\mathbf{r}-1}\leq\frac{C^pt^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}}$$

□

We can integrate now the condition $\Delta_{\mathfrak{p}_i}(Z_{\rho_i})$. The particles in Z_{ρ_i} have to form a distance cluster. Thus every particle is in a ball of radius $|\rho_i|\delta\mathbb{V}$ in $\Lambda^{|\rho_i|-1}$. Because clusters are of size at most γ,

$$\int_{\Lambda^{|\rho_i|-1}\times(\mathbb{R}^d)^{|\rho_i|}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})\tfrac{e^{-\frac14\|V_{\rho_i}\|^2}}{(2\pi)^{d|\rho_i|/2}}d\tilde X_idV_{\rho_i}\leq C_\gamma\mu_\varepsilon^{-|\rho_i|+1}\left(\delta^d\mathbb{V}^d\mu_\varepsilon\right)^{|\rho_i|-1}.$$

In addition, for at least one ρ_i, the set $\underline{\varpi}^i$ is not empty. So we can apply estimate (2.6) and combining the two estimates

$$\begin{aligned}\int\mathbb{1}_{\mathcal{R}^{\rho,\mathfrak{p}}_{((s_i,\bar s_i),(\kappa_i)}}&(Z_p)\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})M^{\otimes p}(V_p)dX_{2,p}dV_p\\ &\leq(\mathbf{r}-1)!\tilde C^p\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}}\prod_{i=1}^{\mathbf{r}}\left(\int\Delta_{\mathfrak{p}_i}(Z_{\rho_i})\tfrac{e^{-\frac14\|V_{\rho_i}\|^2}}{(2\pi)^{d|\rho_i|/2}}d\tilde X_idV_{\rho_i}\right)\\ &\leq(\mathbf{r}-1)!C^p\frac{t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}}\left(\frac{\delta^d\mathbb{V}^d\mu_\varepsilon}{\mu_\varepsilon}\right)^{(\sum_{i=1}^{\mathbf{r}}|\rho_i|-1)-2}\left(\frac{\delta}{\mu_\varepsilon}\right)^2\varepsilon^\alpha.\end{aligned}$$

Every particle annihilated in the time interval $[0,\delta]$ has a clustering collision in this interval and thus is in a distance interval. Therefore $\sum_{i=1}^{\mathbf{r}}(|\rho_i|-1)$ is bigger than $p-n'$. In addition we have chosen θ bigger than $\delta^d\mathbb{V}^d\mu_\varepsilon$ (which is a power of ε) and

$$\int \mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{((s_i,\bar{s}_i),(\kappa_i)}}(Z_p)\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})M^{\otimes p}dX_{2,p}dV_p \leq (\mathbf{r}-1)!\frac{C^p}{\mu_\varepsilon^{p-1}}t^{n_k-1}\theta^{(p-n_k-2)_+}\delta^2\varepsilon^\alpha. \tag{6.26}$$

We sum now on the parameters $((s_i,\bar{s}_i),(\kappa_j))$ and $(\mathfrak{p}_i)$. Because size of $\gamma\delta\mathbb{V}$-distance clusters are bounded by γ, the $|\mathfrak{P}(\rho_i)|$ are smaller than some $C_\gamma>0$ depending only on γ. The conditioning bounds also the number of collision parameters $((s_i,\bar{s}_i),(\kappa_j))$ by $(4\gamma)^{n''}$. Thus

$$\int \left|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\right|M^{\otimes p}dX_{2,p}dV_p \leq \frac{\|h\|(CC_\gamma 4\gamma)^p}{p!\mu_\varepsilon^{p-1}}t^{n_k-1}\theta^{(p-n_k-2)_+}\delta^2\varepsilon^\alpha\sum_{r=1}^{p}\sum_{\underline{\rho}\in\mathcal{P}_p^r}(\mathbf{r}-1)!$$

$$\frac{1}{p!}\sum_{r=1}^{p}\sum_{\underline{\rho}\in\mathcal{P}_p^r}(\mathbf{r}-1)! = \frac{1}{p!}\sum_{\mathbf{r}=1}^{p}\sum_{\substack{k_1+\cdots+k_{\mathbf{r}}=p\\ k_i\geq 1}}\frac{p!}{k_1!\cdots k_{\mathbf{r}}!}\frac{(\mathbf{r}-1)!}{\mathbf{r}!}\leq \sum_{\mathbf{r}=1}^{p}\sum_{\substack{k_1+\cdots+k_{\mathbf{r}}=p\\ k_i\geq 1}}\frac{1}{k_1!\cdots k_{\mathbf{r}}!}\leq e^p$$

hence

$$\int \left|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\right|M^{\otimes p}dX_{2,p}dV_p \leq \frac{\|h\|p\big(e\tilde{C}\big)^p}{\mu_\varepsilon^{p-1}}t^{k-1}\theta^{(p-n_k-2)_+}\delta^2\varepsilon^\alpha\ .$$

This ends the proof of the first inequality.

$\square$

Proof of (6.14). We begin applying (6.21) to bound $|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\Phi^{k'}_{\underline{n},n',n'',p}(Z_m,Z_{p+1,2p-m})|$:

$$\begin{aligned}
&|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\Phi^{k'}_{\underline{n},n',n'',p}(Z_{p+1-m,2p-m})| \\
&\leq \frac{\|h\|^2}{(p!)^2}\sum_{\substack{((s_i,\bar{s}_i),(\kappa_j))\\((s_i,\bar{s}_i),(\kappa_j))}}\ \sum_{\substack{\omega_1\sqcup\omega_2\sqcup\omega_3=[p]\\ \omega_1'\sqcup\omega_2'\sqcup\omega_3'=[p+1-m,2p-m]\\ |\omega_1|=|\omega_1'|=n'\\ |\omega_2|=|\omega_2'|=p-n''}}\ \sum_{\mathbf{p},\mathbf{p}'\geq 1}\ \sum_{\substack{\underline{\varpi}\in\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}\\ \underline{\varpi}'\in\mathcal{Q}^{\mathbf{p}'}_{\omega_1',\omega_2'}}}\mathbb{1}_{\mathcal{R}_{((s_i,\bar{s}_i),(\kappa_j))}}(Z_{\omega_1\cup\omega_3}) \\
&\qquad\times\mathbb{1}_{\mathcal{R}_{((s_i',\bar{s}_i'),(\kappa_j))}}(Z_{\omega_1'\cup\omega_3'})\prod_{i=1}^{\mathbf{p}}\chi(Z_{\varpi_i})\prod_{i=1}^{\mathbf{p}'}\chi(Z_{\varpi_i'}).
\end{aligned}\tag{6.27}$$

Note that the right hand side is invariant under translation. Thus one can fix $x_1=0$ and integrate with respect to the other variables.

For a position Z_{2p-m}, we consider $\underline{\rho}:=(\rho_1,\cdots,\rho_{\mathbf{r}})$ the $\gamma\delta\mathbb{V}$-cluster. We can then construct the parameters $\mathfrak{p}_i:=(\underline{\omega}^i,\underline{\omega}'^i,\underline{\varpi}^i,\underline{\varpi}'^i)$:

- $\underline{\omega}^i:=(\omega_1^i,\omega_2^i,\omega_3^i)$ is a partition of $\rho_i\cap[1,p]$ defined by $\underline{\omega}^i_j:=\omega_j\cap\rho_i$,
- $\underline{\omega}'^i:=(\omega_1^i,\omega_2^i,\omega_3^i)$ is a partition of $\rho_i\cap[p+1-m,2p+m]$ defined by $\underline{\omega}^i_j:=\omega_j\cap\rho_i$,
- $\underline{\varpi}^i:=\{\varpi_j$ such that $\varpi_j\subset\rho_i\}$ and
- $\underline{\varpi}'^i:=\{\varpi_j'$ such that $\varpi_j'\subset\rho_i\}$.

We denote now $\mathfrak{P}(\rho_i)$ the new set of possible parameters $\mathfrak{p}_i$ (this will not create a conflict with the previous section). Because each cluster ρ_i is of size at most γ, $|\mathfrak{P}(\rho_i)|$ is bounded by some constant

C_γ depending only on γ. We define

$$\Delta_{\mathfrak{p}_i}(Z_{\rho_i}) := \mathbb{1}_{Z_{\rho_i}\text{ form a distance cluster}} \prod_{j=1}^{|\underline{\varpi}^i|} \chi(Z_{\varpi^i_j}) \prod_{j=1}^{|\underline{\varpi}'^i|} \chi(Z_{\varpi'^i_j}) \text{ and}$$

$$\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{\substack{((s_i,\bar{s}_i),(\kappa_i))\\((s'_i,\bar{s}'_i),(\kappa'_i))}} := \Big\{ Z_{2p-m}\in\mathcal{D}^{\underline{\rho}}_{\varepsilon},\ Z_{\omega_1\cup\omega_3}\in\mathcal{R}_{((s_i,\bar{s}_i),(\kappa_i)},\ Z_{\omega'_1\cup\omega'_3}\in\mathcal{R}_{((s'_i,\bar{s}'_i),(\kappa'_i)}\Big\}$$

and we have as in the previous case

$$\begin{aligned}&\big|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\Phi^{k'}_{\underline{n},n',n'',p}(Z_m,Z_{p+1,2p-m})\big|\\ &\qquad\leq \frac{\|h\|^2}{(p!)^2}\sum_{\mathbf{r}=1}^{2p-m}\sum_{\underline{\rho}\in\mathcal{P}^r_p}\sum_{\substack{((s_i,\bar{s}_i),(\kappa_j))\\((s'_i,\bar{s}'_i),(\kappa'_j))\\ \underline{\mathfrak{p}}\in\prod_i\mathfrak{P}(\rho_i)}} \mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{\substack{((s_i,\bar{s}_i),(\kappa_i)\\((s'_i,\bar{s}'_i),(\kappa'_i)}}}(Z_{2p-m})\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i}).\end{aligned}$$

Note that, for at least one i, $\underline{\varpi}^i$ is not empty.

As in the previous section, we introduce the change of variable

$$X_{2,(2p-m)}\to((x_{\min\rho_i})_{2\leq i\leq\mathbf{r}},(\bar{X}_i)_{i\leq\mathbf{r}}),\ \ \bar{X}_i:=(x_j-x_{\min\rho_i})_{j\in\rho_i}.$$

Lemma 6.5. *Fixing $\underline{\mathfrak{p}}$, $\underline{\rho}$, $((s_i,\bar{s}_i)_i,\kappa_j)$ and the coordinates V_p and $(\bar{X}_i)_{i\leq\mathbf{r}}$, one obtains the following bound:*

$$\int \mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{\substack{((s_i,\bar{s}_i),(\kappa_i)\\((s'_i,\bar{s}'_i),(\kappa'_i)}}} e^{-\frac{1}{4}\|V_{2p-m}\|^2}dx_{\min\rho_2}\cdots d\hat{x}_{\min\rho_{\mathbf{r}}}\leq\frac{C^p t^{n_k\wedge\mathbf{r}-1}\theta^{(\mathbf{r}-n_k)_+}}{\mu_\varepsilon^{\mathbf{r}-1}} \tag{6.28}$$

Proof. We construct now a clustering tree in order to estimate $\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{\substack{((s_i,\bar{s}_i),(\kappa_i)\\((s'_i,\bar{s}'_i),(\kappa'_i)}}$.

Consider the collision graph associated with the first pseudotrajectory $\mathcal{G}^{[0,t-t_s]}_{\omega_1\cup\omega_3}$ and the graph associated with second one $\mathcal{G}^{[0,t-t_s]}_{\omega'_1\cup\omega'_3}$. Merge them and identify vertices in a same cluster ρ_i. Keeping only the first clustering collisions, we obtain the oriented tree $T^> := (\nu_i,\bar{\nu}_i)_{1\leq i\leq\mathbf{r}-1}$. Note that these clustering collisions can happen in the first or in the second pseudotrajectory.

As in the proof of (6.13) we have to bound the number of collisions of $T^>$ in the time interval $[0,2\tau]$. There are at most $(n_k-1+p-m)$ collision in $[(k'+1)\delta,t-t_s]$ (n_k-1 for the first pseudotrajectory and we have to connect $p-m$ particles in the second). Thus there are at least $(\mathbf{r}-(n_k-1+p-m))_+$ clustering collisions in $[\delta,(k'+1)\delta]\subset[0,2\tau]$.

We explain quickly how to estimate the i-th collision. As in the previous paragraph, we construct the modified tree parameters $(\nu_{(i)},\bar{\nu}_{(i)})$ and the change of variable

$$\forall i\in\{1,\cdots,\mathbf{r}-1\},\ \hat{x}_i:=x_{\min\nu_{(i)}}-x_{\min\bar{\nu}_{(i)}},\ \tilde{X}_i:=(x_j-x_{\min\rho_i})_{j\in\rho_i},$$

$$X_{2,l}\mapsto(\hat{x}_1\cdots,\hat{x}_{\mathbf{r}-1},\tilde{X}_1,\cdots,\tilde{X}_r),$$

and we integrate on the $(\hat{x}_i)_i$.

The clustering set B_i is defined as follows: fix t_{i+1} the time of the $(i+1)$-th clustering collision and the relative positions $\hat{x}_{i-1},\cdots,\hat{x}_1$. We define the i-th clustering set

$$B_i := \bigcup_{\substack{q\in\bigcup_{j\in\nu_{(i)}}\rho_j \\ \bar{q}\in\bigcup_{\bar{j}\in\bar{\nu}_{(i)}}\rho_{\bar{j}}}} \left(B_i^{q,\bar{q}}\cup B_i'^{q,\bar{q}}\right)$$

with

$$B_i^{q,\bar{q}} := \left\{\hat{x}_i \;\middle|\; \exists t_i\in[0,t_{i+1}\wedge T_i],\ |\mathrm{x}_{\bar{q}}(t_i)-\mathrm{x}_{\bar{q}}(t_i)|=\varepsilon\right\},$$

where $\mathrm{x}_i(\tau)$ is the pseudotrajectory with respect to parameters $((s_i,\bar{s}_i)_i,(\kappa_j)_j)$ and $T_i:=2\theta$ for the $(\mathbf{r}-n_k)_+$ first collisions, t else, and $B_i'^{q,q'}$ is defined in the same way for the other pseudotrajectory. We can apply the estimate of the previous paragraph:

$$\int \mathbb{1}_{B_i}d\hat{x}_i \leq \frac{2C}{\mu_\varepsilon}\sum_{\substack{\nu_i\in\nu_{(i)}\\ \bar{\nu}_i\in\bar{\nu}_{(i)}}}\left(|\rho_{\nu_i}|+\|V_{\rho_{\nu_i}}\|^2\right)\left(|\rho_{\bar{\nu}_i}|+\|V_{\rho_{\bar{\nu}_i}}\|^2\right)\int_0^{t_{i+1}\wedge T_i}dt_i.$$

In this way, we end up with the same situation as in the proof of Lemma 6.4 and we can apply the same argument. □

Lemma 6.5 provides a similar estimation than Lemma 6.4. In a second time, one can integrate the $\Delta_{\mathfrak{p}_i}$ with respect to $(V_p,(\bar{X}_i)_i)$ using the same computation as for the proof of (6.13),

$$\begin{aligned}\int \left|\Phi^{k'}_{\underline{n},n',n'',p}(Z_p)\Phi^{k'}_{\underline{n},n',n'',p}(Z_{p+1-m,2p-m})\right|M^{\otimes(2p-m)}dX_{2,2p-m}dV_{2p-m}&\\ \leq \frac{(2p-m)!\|h\|^2}{(p!)^2\mu_\varepsilon^{2p-m-1}}C^p\delta^2\varepsilon^\alpha\tau^{(p-n_k-2)_+}t^{n_k-1+p-m}&\\ \leq \frac{\|h\|^2}{p^m\mu_\varepsilon^{2p-m-1}}\tilde{C}^p\delta^2\varepsilon^\alpha\tau^{(p-n_k-2)_+}t^{n_k-1+p-m}&\end{aligned}$$

which concludes the proof. □

Proof of (6.15). In $\mathsf{f}_{p\leftarrow l}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\right]\left(Z_{[1,l]}\right)\mathfrak{X}_{n',p}(Z_{[1,p]})$ we have three set of indices:

- $[1,n']$ the set of particles created in the final pseudotrajectory,
- $[n'+1,p]$ the particles added in the treatment of local recollision and
- $[p+1,l]$ particles added in the dynamical cluster development.

Any permutation σ which sends $[1,n']$, $[n'+1,p]$ and $[p+1,l]$ onto themselves stabilizes

$$\mathsf{f}_{p\leftarrow l}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\otimes\mathbb{1}^{\otimes(p-n')}\right]\left(Z_{[1,l]}\right)\mathfrak{X}_{n',p}(Z_{[1,p]})$$

and

$$\Phi^{k'}_{\underline{n},n',p,l}(Z_l)\frac{n'!\,(p-n')!\,(l-p)!}{l!}\sum_{\substack{\omega_1\sqcup\omega_2\sqcup\omega_3=[l]\\ |\omega_1|=n'\\ |\omega_2|=l-p}}\mathsf{f}_{p\leftarrow l}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\otimes\mathbb{1}^{\otimes(p-n')}\right](Z_{\omega_1\cup\omega_2},Z_{\omega_3})\times\mathfrak{X}_{n',p}(Z_{\omega_1},Z_{\omega_2})$$

We develop $\mathsf{f}_{p\leftarrow l}\left[\Phi^{0,k'}_{\underline{n},n'}[h]\otimes\mathbb{1}^{\otimes(p-n')}\right]$: for $\underline{\omega}=(\omega_1,\omega_2,\omega_3)$, $\underline{\lambda}$ two partitions of $[1,l]$ with $\omega_1\cup\omega_2\subset\lambda_1$ and $(s_i,\bar{s}_i)_{1\leq i\leq n'-1}$, we define $\mathcal{R}^{\underline{\omega},\underline{\lambda}}_{(s_i,\bar{s}_i)}\subset\mathcal{D}^l_\varepsilon$ the set of initial data such that particles in λ_1 form a $(\omega_1\cup\omega_2)$-cluster (see the previous part for the definition of cluster pseudotrajectories), and the tree pseudotrajectory $\mathrm{Z}_{n'}(\tau,(s_i,\bar s_i),\mathsf{Z}^{\lambda_1}_{\omega_1}(\delta))$ with $\sum_{i=1}^j k_i$ particles at time $\tau:=(k'+1)\delta+(k-j)\tau$. Then we can write:

$$\Phi^{k'}_{\underline{n},n',p,l}(Z_l)$$
$$=-\frac{1}{l!}\sum_{\substack{\omega_1\sqcup\omega_2\sqcup\omega_3=[l]\\|\omega_1|=n'\\|\omega_2|=p-n'}}\sum_{\mathbf{l}=1}^{l}\sum_{\substack{\lambda_1\subset[l]\\\omega_1\cup\omega_2\subset\lambda_1}}\sum_{\substack{(\lambda_2,\cdots,\lambda_{\mathbf{l}})\\\in\mathcal{P}^{\mathbf{l}-1}_{\lambda_1^c}}}\sum_{(s_i,\bar s_i)}\sum_{\substack{\mathbf{p}\geq1\\\underline{\varpi}\in\mathcal{Q}^{\mathbf{p}}_{\omega_1,\omega_2}}}\mathbb{1}_{\mathcal{R}^{\underline{\omega},\underline{\lambda}}_{(s_i,\bar s_i)}}h(\mathrm{Z}_{n'}(\tau,(s_i,\bar s_i),\mathsf{Z}^{\lambda_1}_{\omega_1}(\delta)))\prod_{i=1}^{\mathbf{p}}\chi(Z_{\varpi_i})$$
$$\times\prod_{i=1}^{n}\bar s_i\prod_{i=2}^{\mathbf{l}}\varphi_{|\lambda_i|}(Z_{\lambda_i})\psi_{\mathbf{l}}(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathbf{l}}}).$$

The pseudotrajectory of the particles is represented in Figure 11.

We recall the Penrose's tree inequality (see for example the second section of [6] for a proof)

$$\left|\psi_{\mathbf{l}}(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathbf{l}}})\right|=\left|\sum_{C\in\mathcal{C}(\omega)}\prod_{(i,j)\in E(C)}-\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\right|\leq\sum_{T\in\mathcal{T}_{\mathbf{l}}}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}.$$

Hence we obtain the following bound on $\Phi^{k'}_{\underline{n},n',p,l}$, invariant under translation:

$$\left|\Phi^{k'}_{\underline{n},n',p,l}\right|(Z_l)\leq\frac{\|h\|}{l!}\sum_{\substack{\underline{\omega}\in\mathcal{P}^3_l\\|\omega_1|=n'\\|\omega_2|=p}}\sum_{\mathbf{l}=1}^{l}\sum_{\substack{\underline{\lambda}\in\mathcal{P}^{\mathbf{l}}_l\\\omega_1\cup\omega_2\subset\lambda_1}}\sum_{(s_i,\bar s_i)}\sum_{\mathbf{p}\geq1}\mathbb{1}_{\mathcal{R}^{\underline{\omega},\underline{\lambda}}_{(s_i,\bar s_i)}}\prod_{i=1}^{\mathbf{p}}\chi(Z_{\varpi_i})\prod_{i=2}^{\mathbf{l}}\varphi_{|\lambda_i|}(Z_{\lambda_i})\times\sum_{T\in\mathcal{T}_{\mathbf{l}}}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}.\tag{6.29}$$

We will use again the distance cluster to control the relations between particles in the time interval $[0,\delta]$. Let $\underline{\rho}:=(\rho_1,\cdots,\rho_{\mathbf{r}})$ the distance partition of Z_l. For each ρ_i, we construct the collision parameter $\mathfrak{p}:=(\underline{\omega}^i,\underline{\lambda}^i,\underline{\varpi}^i)$ with:

- $\underline{\omega}^i:=(\omega^i_1,\omega^i_2,\omega^i_3)$ is a partition of $\rho_i\cap[1,p]$ defined by $\underline{\omega}^i_j:=\omega_j\cap\rho_i$,
- $\underline{\lambda}^i:=\{\lambda^i_1:=\lambda_1\cap\rho_i\}\cup\{\lambda_j$ for $j\geq2$ with $\lambda_j\subset\rho_i\}$ a partition of ρ^i and
- $\underline{\varpi}^i:=\{\varpi_j$ such that $\varpi_j\subset\rho_i\}$,

and we denote $\mathfrak{P}(\rho_i)$ the set of possible $\mathfrak{p}_i$.

The global conditioning bounds velocities, so that particles which form a collisional cluster have to be in a same distance cluster. Thus for each ϖ_j and λ_k, $k\geq2$ there exists a ρ_i containing λ_k or ϖ_j. In addition, for $i\neq i'$, particles in λ^i_1 do not interact with particles of $\lambda^{i'}_1$. The overlaps are also contained in the distance cluster: if we denote two dynamical clusters λ_j and $\lambda_{j'}$ with $j,j'\geq2$, there exists a ρ_i containing both, and if $\lambda_j\subset\rho_i$ has an overlap with λ_1, then λ_j has an overlap with λ^i_1. This last property allows us to rewrite the overlap cumulant: on $\mathcal{D}^{\underline{\rho}}_{\varepsilon}$,

$$\left|\psi_{\mathbf{l}}(Z_{\lambda_1},\cdots,Z_{\lambda_{\mathbf{l}}})\right|\leq\sum_{T\in\mathcal{T}_{\mathbf{l}}}\prod_{(i,j)\in E(T)}\mathbb{1}_{\lambda_i\overset{\circ}{\sim}\lambda_j}\leq\prod_{i=1}^{\mathbf{r}}\sum_{T_i\in\mathcal{T}_{|\rho_i|}}\prod_{(j,j')\in E(T_i)}\mathbb{1}_{\lambda^i_j\overset{\circ}{\sim}\lambda^i_{j'}}\leq\prod_{i=1}^{\mathbf{l}}\left|\mathcal{T}(\rho_i)\right|.$$

We have now the following bound

$$\left|\Phi^{k'}_{\underline{n},n',p,l}(Z_l)\right|\leq\frac{\|h\|}{l!}\sum_{\mathbf{r}=1}^{l}\sum_{\underline{\rho}\in\mathcal{P}^{\mathbf{r}}_l}\sum_{\substack{(s_i,\bar s_i)\\\underline{\mathfrak{p}}\in\prod_i\mathfrak{P}(\rho_i)}}\mathbb{1}_{\mathcal{R}^{\underline{\rho},\underline{\mathfrak{p}}}_{(s_i,\bar s_i)}}\prod_{i=1}^{\mathbf{r}}\Delta_{\mathfrak{p}_i}(Z_{\rho_i})$$

with

$$\Delta_{\mathfrak{p}_i}(Z_{\rho_i}) := \Big|\mathcal{T}(\rho_i)\Big| \mathbb{1}_{Z_{\rho_i}\text{ form a distance cluster}} \prod_{j=1}^{|\varpi^i|} \chi(Z_{\varpi^i_j}) \text{ and}$$

$$\mathcal{R}^{\rho,\mathfrak{p}}_{(s_i,\bar{s}_i)} := \Big\{ Z_l \in \mathcal{D}^{\rho}_{\varepsilon},\, Z_l \in \mathcal{R}^{\omega,\lambda}_{(s_i,\bar{s}_i)} \Big\}.$$

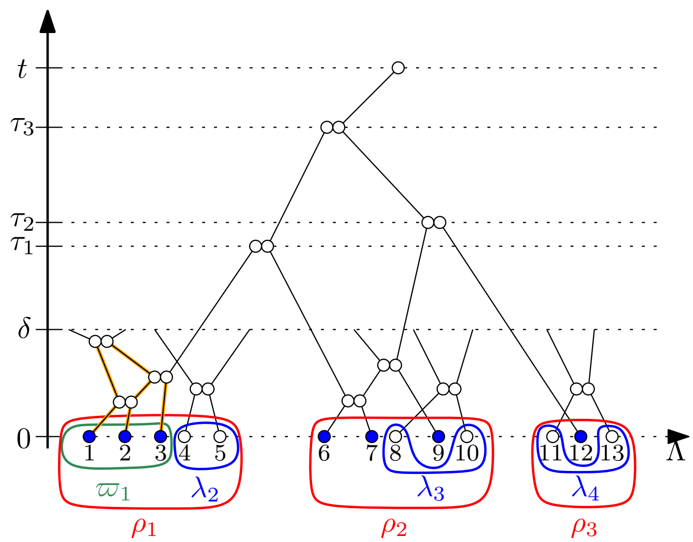

FIGURE 11. Example of construction of a clustering tree. Here, $\omega_1 = \{3, 6, 9, 12\}$, $\omega_2 = \{1, 2\}$ and $\omega_3 = (4, 5, 7, 8, 10, 11, 13)$. In addition the set λ_1 is equal to $\{1, 2, 3, 6, 7, 9, 12\}$ (represented by the blue particles). The particles of ϖ_1 are involved in a pseudotrajectory (represented in orange) leading to a recollision. The black pseudotrajectory represents the trajectory of the last existing particle (here 9). We can construct a clustering tree as in the proof of (6.13): $T_> := ((1, 2), (2, 3))$.

Finally we construct the clustering tree : we consider the collision graph of the particles ω_1 on the time interval $[\delta, t - t_s]$. Then we identify vertices in a same cluster ρ_i and we keep only the first clustering collision. This constructs an ordered tree $T^> \in \mathcal{T}^>_r$.

As in the previous cases, the condition of respecting the collision history $T^>$ depends only on the relative positions at time δ which are the same than at time 0 (clusters do not interact). We can apply the same method than in the estimation (6.13) and we obtain the expected bound. □

Proof of (6.16). We adapt the proof of (6.14) with the parametrisation of the previous part. □

7. Treatment of the principal part

In this section we conclude the proof of our main theorem (Equation (1.14)), by discussing the main term of the expansion $G^{\text{main}}_{\varepsilon}(t)$.

7.1. **Duality formula.** We recall that

$$G_\varepsilon^{\mathrm{main}}(t) = \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1/2}\sum_{(i_1,\cdots,i_{n_K})}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}}(0)\right)\zeta^0_\varepsilon(g)\right] = \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\mu_\varepsilon^{n_k}\,\hat\Phi^0_{\underline{n}}[h]\,\hat g\right]$$

where $\Phi^0_{\underline{n}}[h]$ is the development of $h(z_i(t))$ along pseudotrajectories with n_k particles at time $t-n_k\theta$ and no recollision.

We denote

$$g_n^\varepsilon(Z_n) := \left(\sum_{k=1}^n g(z_k)\right)\ \frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\geq 0}\frac{\mu_\varepsilon^p}{p!}\int e^{-\mathcal{V}_{n+p}(X_{n+p})}dX_{n+1,n+p}. \tag{7.1}$$

Then using the equality (3.7) and L^1 estimations on $\Phi^0_{\underline{n}}$ of Section 4, we have for h and g in L^∞

$$G_\varepsilon^{\mathrm{main}}(t) = \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} \mathbb{E}_\varepsilon\left[\mu_\varepsilon^{-1}\sum_{\underline{i}_{n_K}}\Phi^0_{\underline{n}}[h]\left(\mathbf{Z}_{\underline{i}_{n_k}}(0)\right)\sum_{j=1}^{n_K} g(\mathbf{z}_{i_j}(0))\right] + O\left(\varepsilon\sum_{\underline{n}}(Ct)^{n_k}\|h\|\|g\|\right)$$

$$= \sum_{\underline{n}}\int \mu_\varepsilon^{n_K-1}\Phi^0_{\underline{n}}[h]\left(Z_{n_K}\right)g^\varepsilon_{n_K}(Z_{n_K})\,\frac{e^{-\mathcal{H}_{n_K}(Z_{n_K})}dZ_{n_K}}{(2\pi)^{\frac{n_K d}{2}}} + O\left(\varepsilon\left(K2^{K^2}(Ct)^{2^{K+1}}\|h\|\|g\|\right)\right).$$

We want to compute the asymptotics of each term in the sum.

$$\int \mu_\varepsilon^{n_K-1}\Phi^0_{\underline{n}}[h]\left(Z_{n_k}(0)\right)g^\varepsilon_{n_k}(Z_{n_k})\,\frac{e^{-\mathcal{H}_{n_K}(Z_{n_K})}dZ_{n_K}}{(2\pi)^{\frac{n_K d}{2}}}$$
$$= \frac{\mu_\varepsilon^{n_K-1}}{n_K!}\int\sum_{(s_i,\bar s_i)_i}\prod_{i=1}^{n_K-1}\bar s_i\mathbb{1}_{\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)}}h\big(\mathrm{Z}_{n_k}(t)\big)g^\varepsilon_{n_K}(Z_{n_K})\,M^{\otimes n_K}dZ_{n_K}$$

where $\mathcal{R}^{\underline{n}}_{(s_i,\bar s_i)}\subset\mathcal{D}_\varepsilon^{n_K}$ is the set of initial parameters such that for each $k\in[0,K]$, the pseudotrajectory $\mathrm{Z}_{n_K}(\tau,(s_i,\bar s_i))$ has n_k particles at time $t-k\theta$, and no recollision.

We order now the annihilations. Fixed an initial position Z_{n_K} and given collision parameters $(s_i,\bar s_i)_i$, we can construct a collision tree (a_i,b_i) where the i-th removed particle is b_i, after a collision with a_i. We have a one-to-one correspondence between the admissible $(a_i,b_i)_i$ and the $(s_i)_i$, thus we can change the collision parameters to $(a_i,b_i,\bar s_i)$. Due to the symmetry of $g^\varepsilon_{n_K}$, we can reorder particles by setting $b_i=n_K-i+1$. Denoting $\tilde a_i:=a_{n_K-i+1}$, $\tilde s_i:=\bar s_{n_K-i+1}$ and $\mathcal{R}^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}$ the set of initial parameters respecting the collision constraints $(\tilde a_i,\tilde s_i)_{2\leq i\leq n_K}$,

$$\int\mu_\varepsilon^{n_K-1}\Phi^0_{\underline{n}}[h]\left(Z_{n_K}\right)g^\varepsilon_{n_K}(Z_{n_K})^{\otimes n_K}dZ_{n_K}$$
$$= \mu_\varepsilon^{n_K-1}\sum_{(\tilde a_i,\tilde s_i)_i}\prod_{i=2}^{n_K}\tilde s_i\int_{\mathcal{R}^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}}h(\mathrm{Z}_{n_K}(t))g^\varepsilon_{n_K}(Z_{n_K})M^{\otimes n_K}dZ_{n_K}.$$

Note that the admissible $(a_i)_{2\leq i\leq n_K}$ verifies $a_i\in[1,i-1]$.

We define now the *backward pseudocharacteristic*

$$\xi^\varepsilon_{n_K}(\tau,(\tilde a_i,\tilde s_i)_i,z_1,(t_i,\bar v_i,\eta_i)_{2\leq i\leq n_K})$$

with a final point z_1 and parameters $(\tau_i, \bar v_i, \eta_i)_{2\le i\le n_K} \in (\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}^{d-1})^{n_K-1}$ with $t > t_2 > \cdots > t_{n_K} > 0$. We construct sequentially the trajectory on each $[t_{i+1}, t_i]$. We begin at time t with particle 1 at z_1. Let us specify the coordinate of the pseudocharacteristic at time $\tau \in (t_i, t_{i-1})$. In the interval (t_{i+1}, t_i), there are $i-1$ particles $\xi^\varepsilon_{n_K}(\tau) = (z^\varepsilon_1(\tau), \cdots, z^\varepsilon_{i-1}(\tau))$ which move along straight line (backwards). At time t_i^+, we add particle i at position $(x^\varepsilon_{\tilde a_i}(\tau) + \varepsilon\eta_i, \bar v_i)$. If $\tilde s_i = 1$ we apply the scattering between particles $\tilde a_i$ and i, else the particles do not interact. Note that the velocities $v^\varepsilon_i(\tau)$ do not depend on ε.

We denote $\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)$ and $\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}(z_1)$ the definition set of pseudocharecteristics: for $z_1 \in \Lambda\times\mathbb{R}^d$

$$\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1) := \Big\{ (t_i, \bar v_i, \eta_i)_i \in (\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^{d-1})^{n_K-1} \Big| \, t > t_2 > \cdots > t_{n_K} > 0,$$
$$\forall i \in [n_{j-1}+1, n_j], t_i \in (t - j\theta, t-(j-1)\theta), \ (v^\varepsilon_i(t_i^+) - \bar v_i)\cdot\eta_i < 0 \Big\},$$

and $\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}(z_1)$ the subset of $\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)$ such that distances between particles are bigger than ε when one particle is created (trajectories without *overlap*).

We then perform do the change of variables

$$\begin{aligned} \bigcup_{z_1\in\mathbb{D}} \{z_1\} \times \mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1) &\longrightarrow \mathcal{R}^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i} \\ (z_1, (t_i, \bar v_i, \eta_i)_i) &\longmapsto \xi^\varepsilon_{n_K}(\tau = 0). \end{aligned} \tag{7.2}$$

Since we have removed all the recollisions, this map is a bijection. It is a local diffeomorphism, hence a diffeomorphism. It sends a measure

$$M(v_1)dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i} := M(v_1)dz_1 \prod_{i=2}^{n_K} \big((v^\varepsilon_{a(i)}(t_i^+) - \bar v_i)\cdot\eta_i\big)_+ M(v_i) d\bar v_i d\eta_i dt_i \tag{7.3}$$

onto $\mu_\varepsilon^{n_k-1} M^{\otimes n_K} dZ_{n_K}$. We will denote with a little abuse of notation:

$$\mathbb{D}\times\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i} := \bigcup_{z_1\in\mathbb{D}} \{z_1\}\times\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}(z_1).$$

We finally arrive to the following duality formula

$$\begin{aligned} &\mu_\varepsilon^{n_K-1} \int \Phi^0_{\underline{n}}[h]\,(Z_{n_K})\, g^\varepsilon_{n_K}(Z_{n_K}) M^{\otimes n_K}(V_{n_K}) dZ_{n_K} \\ &\qquad = \sum_{(\tilde a_i,\tilde s_i)_i} \prod_{i=1}^{n_K-1} \tilde s_i \int_{\mathbb{D}\times\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}} h(z_1) g^\varepsilon_{n_K}(\xi^\varepsilon_{n_K}(0)) M(v_1) dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}. \end{aligned} \tag{7.4}$$

Denoting

$$g_{n_K}(Z_{n_K}) := \sum_{i=1}^{n_K} g(z_i),$$

we have formally

$$\begin{aligned} &\int \mu_\varepsilon^{n_K-1} \Phi^0_{\underline{n}}\,(Z_{n_K})\, g^\varepsilon_{n_K}(Z_{n_K}) M^{\otimes n_K}(V_{n_K}) dZ_{n_K} \\ &\qquad \underset{\varepsilon\to0}{\longrightarrow} \sum_{(\tilde a_i,\tilde s_i)_i} \prod_{i=1}^{n_K-1} \tilde s_i \int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}} h(z_1) g_{n_K}(\xi^0_{n_K}) M^{\otimes n_K}(V_{n_K}) dZ_{n_K} d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}. \end{aligned}$$

In order to have explicit rates of convergence, we decompose the error in three parts:

$$\begin{aligned}
&\int \mu_\varepsilon^{n_K-1}\,\Phi^0_{\underline{n}}\, g^\varepsilon_{n_K} M^{\otimes n_K}(V_{n_K}) dZ_{n_K}\\
&= \sum_{(\tilde a_i,\tilde s_i)_i}\prod_{i=1}^{n_K-1}\tilde s_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}} h(z_1) g_{n_K}(\xi^0_{n_K}(0)) M(v_1) dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i} + R_1+R_2+R_3
\end{aligned} \tag{7.5}$$

$$R_1 = \sum_{(\tilde a_i,\tilde s_i)_i}\prod_{i=1}^{n_K-1}\tilde s_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}} h(z_1)\left(g_{n_K}(\xi^\varepsilon_{n_K}(0)) - g_{n_K}(\xi^0_{n_K}(0))\right) M(v_1) dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}$$

$$R_2 = -\sum_{(\tilde a_i,\tilde s_i)_i}\prod_{i=1}^{n_K-1}\tilde s_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}} h(z_1) g_{n_K}(\xi^\varepsilon_{n_K}(0))\left(1-\mathbb{1}_{\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}(z_1)}\right) M(v_1) dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}$$

$$R_3 = \sum_{(\tilde a_i,\tilde s_i)_i}\prod_{i=1}^{n_K-1}\tilde s_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}} h(z_1)\left(g^\varepsilon_{n_K}(\xi^\varepsilon_{n_K}(0)) - g_{n_K}(\xi^\varepsilon_{n_K}(0))\right) M(v_1) dz_1 d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}.$$

They are estimated using the following standard results:

Lemma 7.1. *Fix $\bar n := (n_1,\cdots,n_k)$. For any $\varepsilon>0$ sufficiently small, we have for $p\in[1,2]$ and $z_1\in\mathbb{D}$*

$$\begin{aligned}
&\sum_{(\tilde a_i,\tilde s_i)_i}\int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)}\left(\prod_{i=2}^{n_K}\left\|v^\varepsilon_{\tilde a_i}(t_i^+)-\bar v_i\right\|^p M(\bar v_i) d\bar v_i d\eta_i dt_i\right)\frac{e^{-\frac12\|v_1\|^2}}{(2\pi)^{d/2}}\\
&\qquad\le (C(K-1)\theta)^{n_{K-1}}(C\theta)^{n_K-n_{K-1}} e^{-\frac{\|v_1\|^2}{4}}\,.
\end{aligned} \tag{7.6}$$

Proof. We follow the proof of Lemma 4.2 in [22].

For $i\in[2,n_K]$ we forget parameters $(\tilde a_j)_{i<j\le n_K}$ and $(t_j,\bar v_j,\eta_j)_{i<j\le n_K}$:

$$\begin{aligned}
&\sum_{\tilde a_i=1}^{i-1}\left\|v^\varepsilon_{\tilde a_i}(t_i^+)-\bar v_i\right\|^p e^{-\frac{\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2}{8n_K}-\frac{\|\bar v_i\|^2}{8}}\\
&\quad\le 2^{p-1}\left[\sum_{j=1}^{i-1}\|v_j(t_i^+)\|^p+(i-1)\|\bar v_i\|^p\right] e^{-\frac{\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2}{8n_K}+\frac{\|\bar v_i\|^2}{8}}\\
&\quad\le 2^{p-1}\left[\left(\sum_{j=1}^{i-1}\|v_j(t_i^+)\|^2\right)^{p/2}(i-1)^{1-p/2}+(i-1)\|\bar v_i\|^p\right] e^{-\frac{\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2}{8n_K}+\frac{\|\bar v_i\|^2}{8}}\,,
\end{aligned}$$

$$\begin{aligned}
&\sum_{\tilde a_i=1}^{i-1}\left\|v^\varepsilon_{\tilde a_i}(t_i^+)-\bar v_i\right\|^p e^{-\frac{\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2}{8n_K}-\frac{\|\bar v_i\|^2}{8}}\\
&\quad\le 2^{p-1}\left[\left(\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2\right)^{p/2}(i-1)^{1-p/2}+(i-1)\|\bar v_i\|^p\right] e^{-\frac{\|v_1\|^2+\sum_{j=2}^{i-1}\|\bar v_j\|^2}{8n_K}+\frac{\|\bar v_i\|^2}{8}}\\
&\quad\le C\left[n_K^{p/2}(i-1)^{1-p/2}+(i-1)\right]\le Cn_K.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{(\tilde a_i,\tilde s_i)_i}\int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)}\left(\prod_{i=2}^{n_K}\left\|v^\varepsilon_{\tilde a_i}(t_i^+)-\bar v_i\right\|^p M(\bar v_i)d\bar v_i d\eta_i dt_i\right)\frac{e^{-\frac12\|v_1\|^2}}{(2\pi)^{d/2}}\\
&\le \int_\theta^{K\theta}dt_2\cdots\int_\theta^{t_{n_{K-1}-1}}dt_{n_{K-1}}\int_0^\theta dt_{n_{K-1}+1}\cdots\int_0^{t_{n_K}-1}dt_{n_K}\\
&\qquad\times\int_{(\mathbb{S}^{d-1}\times\mathbb{R}^d)^{n_K-1}}C^{n_K-1}n_K^{n_K-1}e^{-\frac{\|v_1\|^2+\Sigma_{j=2}^{i-1}\|\bar v_j\|^2}{4}}\left(\prod_{i=2}^{n_K}\frac{d\bar v_i d\eta_i}{(2\pi)^{d/2}}\right)\\
&\le\frac{C(C(K-1)(n_K-1)\theta)^{n_{K-1}-1}}{(n_{K-1}-1)!}\frac{(C\theta n_K)^{n_K-n_{K-1}}}{(n_K-n_{K-1})!}\le(\tilde Ct)^{n_K}.
\end{aligned}$$

□

Lemma 7.2. *Fix $\bar n := (n_1,\cdots,n_K)$. For any $\varepsilon>0$ sufficiently small, we have*

$$\sum_{(\tilde a_i,\tilde s_i)_i}\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}}\left|1-\mathbb{1}_{\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde a_i,\tilde s_i)_i}(z_1)}\right|M(v_1)dz_1d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i}\le(Ct)^{n_K}\varepsilon^\alpha\ . \tag{7.7}$$

This is an estimation of the set of parameters leading to an overlap. It can be obtained in the same as the estimation of recollisions of Section 3.

From Lemma 7.2 we deduce

$$|R_1|\le C(Ct)^{n_K}\varepsilon^\alpha\|g\|\,\|h\|.$$

Lemma 7.3. *Fix $\bar n := (n_1,\cdots,n_k)$, $\varepsilon>0$ sufficiently small, and $X_{n_K}\in\Lambda^{n_K}$ such that for $i\neq j$*

$$|x_i-x_j|>\varepsilon.$$

Then

$$\left|1-\frac{1}{\mathcal{Z}_\varepsilon}\sum_{p\ge0}\frac{\mu_\varepsilon^p}{p!}\int e^{-\mathcal{V}_{n_K+p}(X_{n_K},\bar X_p)}d\bar X_p\right|\le C^{n_K}\varepsilon\ . \tag{7.8}$$

Proof. Using the formula (3.9), for any $X_{n_K}\in\Lambda^{n_K}$ with $|x_i-x_j|>\varepsilon$ for $i\neq j$,

$$\begin{aligned}
\exp\left(-\mathcal{V}^\varepsilon_{n_K+p}(X_{n_K},\underline{X}_p)\right)&=\sum_{\omega\subset[1,p]}e^{-\mathcal{V}^\varepsilon_{n_K}(X_{n_K})-\mathcal{V}^\varepsilon_{|\omega^c|}(\underline{X}_{\omega^c})}\psi_p^{n_K}(X_{n_K},\underline{X}_\omega)\\
&=\sum_{\omega\subset[1,p]}e^{-\mathcal{V}^\varepsilon_{|\omega^c|}(\underline{X}_{\omega^c})}\psi_p^{n_K}(X_{n_K},\underline{X}_\omega).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{p\ge0}\frac{\mu_\varepsilon^p}{p!}\int e^{-\mathcal{V}_{n_K+p}(X_{n_K},\bar X_p)}d\bar X_p&=\sum_{p\ge0}\sum_{p_1+p_2=p}\frac{\mu_\varepsilon^p}{p!}\frac{p!}{p_1!p_2!}\int e^{-\mathcal{V}^\varepsilon_{p_2}(\underline{X}'_{p_2})}\psi^{n_K}_{p_1}(X_{n_K},\underline{X}_{p_1})d\bar X_{p_1}d\bar X'_{p_2}\\
&=\mathcal{Z}_\varepsilon\sum_{p\ge0}\frac{\mu_\varepsilon^p}{p!}\int\psi_p^{n_K}(X_{n_K},\underline{X}_p)d\bar X_p\\
&=\mathcal{Z}_\varepsilon\left(1+\sum_{p\ge1}\frac{\mu_\varepsilon^p}{p!}\int\psi_p^{n_K}(X_{n_K},\underline{X}_p)d\bar X_p\right).
\end{aligned}$$

Using the estimation (3.12),

$$\sum_{p\geq 1}\frac{\mu_\varepsilon^p}{p!}\int \psi_p^{n_K}(X_{n_K},\underline{X}_p)d\bar{X}_p \leq \sum_{p\geq 1}\frac{\mu_\varepsilon^p}{p!}(p-1)!\big(Ce\varepsilon^d\big)^p n_K e^{n_K} \leq \sum_{p\geq 1}\big(C'\varepsilon\big)^p n_K e^{n_K} \leq 2\varepsilon n_K e^{n_K}$$

for ε small enough. This concludes the proof. □

Using Lemmata 7.1 and 7.3 we obtain

$$|R_3| = C(Ct)^{n_K}\varepsilon\|g\|\,\|h\|.$$

Lemma 7.4. *Fix $\bar{n} := (n_1,\cdots,n_k)$, $\varepsilon > 0$ and $(z_1,(t_i,\bar{v}_i,\eta_i)_i) \in \mathbb{D}\times\mathbb{G}^{\underline{n},\varepsilon}_{(\tilde{a}_i,\tilde{s}_i)_i}$. We have*

$$\big|\xi^\varepsilon_{n_K}(0)-\xi^0_{n_K}(0)\big| \leq n_K^{3/2}\varepsilon\ . \tag{7.9}$$

Proof. We recall first that the two trajectories $\xi^\varepsilon_{n_K}(\tau)$ and $\xi^0_{n_K}(\tau)$ have coincident velocities and at each annihilation of a particle there is a new shift of size ε. Thus for any i bigger than 1, $\|x_i^\varepsilon(\tau)-x_i^0(\tau)\| \leq (i-1)\varepsilon$ and, summing up,

$$\|\xi^\varepsilon_{n_K}(\tau)-\xi^0_{n_K}(\tau)\|^2 \leq n_K^3\varepsilon^2.$$

□

If g is uniformly Lipschitz, by Lemmata 7.1 and 7.4 we get

$$|R_1| = C(Ct)^{n_K}\varepsilon\|\nabla g\|\,\|h\|.$$

Finally we get for h and g Lipschitz

$$\int \mu_\varepsilon^{n_K-1}\,\Phi^0_{\underline{n}}\,g^\varepsilon_{n_K}M^{\otimes n_K}dZ_{n_K} = \sum_{(\tilde{a}_i,\tilde{s}_i)_i}\prod_{i=1}^{n_K-1}\tilde{s}_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde{a}_i,\tilde{s}_i)_i}} h(z_1)g_{n_K}(\xi^0_{n_K}(0))M(v_1)dz_1d\Lambda^{\underline{n}}_{(\tilde{a}_i,\tilde{s}_i)_i}$$
$$+O\Big(\varepsilon^\alpha(Ct)^{n_K}\|h\|\big(\|g\|+\|\nabla g\|\big)\Big).$$

and therefore

$$\begin{aligned} G^{\mathrm{main}}_\varepsilon(t) = &\sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}}\sum_{(\tilde{a}_i,\tilde{s}_i)_i}\prod_{i=1}^{n_K-1}\tilde{s}_i\int_{\mathbb{D}\times\mathbb{G}^{\underline{n},0}_{(\tilde{a}_i,\tilde{s}_i)_i}} h(z_1)g_{n_K}(\xi^0_{n_K})M(v_1)dz_1d\Lambda^{\underline{n}}_{(\tilde{a}_i,\tilde{s}_i)_i}\\ &+O\left(\varepsilon^\alpha K2^{K^2}(Ct)^{2^{K+1}}\|h\|\big(\|g\|+\|\nabla g\|\big)\right). \end{aligned} \tag{7.10}$$

7.2. **Linearized Boltzmann equation.** Let $\mathbf{g}(t)$ be the solution of the linearized Boltzmann equation:

$$\partial_t\mathbf{g}(t)+v\cdot\nabla_x f(t) = \mathcal{L}\mathbf{g}(t)\text{ for }(t,x,v)\in[0,\infty)\times\mathbb{D}$$
$$\mathbf{g}(t=0) = g\text{ on }\mathbb{D},$$

and $\mathcal{L}$ the linearized Boltzmann operator:

$$\mathcal{L}g(v) := \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}\big(g(v')+g(\bar{v}')-g(v)-g(\bar{v})\big)((v-\bar{v})\cdot\eta)_+M(\bar{v})d\eta\,d\bar{v}$$

with $(v',\bar{v}')$ defined by the scattering rule (1.11).

We can write this equation in Duhamel form: denoting $S(\tau)$ the semigroup associated with $v\cdot\nabla_x$,

$$\mathbf{g}(t) = S(t)g + \int_0^t S(t-\tau_1)\mathcal{L}\mathbf{g}(\tau_1)d\tau_1.$$

We want to iterate this formula, but cutting trees with superexponential growth of the number of annihilation (as in the hard sphere system): defining

$$Q_{m,n}(\tau)[g] = \int_0^\tau dt_{m+1}\int_0^{t_{m+1}}\cdots\int_0^{t_{n-1}} dt_n S(t-t_{m+1})\mathcal{L}S(t_{m+1}-t_{m+2})\cdots\mathcal{L}S(t_n)g,$$

and for $\underline{n} := (n_1,\cdots,n_k)$ with $1\leq n_1\leq\cdots\leq n_k$,

$$Q_{\underline{n}}(\tau)g = Q_{1,n_1}(\tfrac{\tau}{k})Q_{n_1,n_2}(\tfrac{\tau}{k})\cdots Q_{n_{k-1},n_k}(\tfrac{\tau}{k})[g],$$

we have

$$\mathbf{g}(t) = \sum_{\substack{n_1\leq\cdots\leq n_K\\ n_j-n_{j-1}\leq 2^j}} Q_{\underline{n}}(t)[g] + \sum_{k=1}^{K}\sum_{\substack{n_1\leq\cdots\leq n_{k-1}\\ n_j-n_{j-1}\leq 2^j}}\sum_{n_k>2^k} Q_{\underline{n}}(k\tau)[\mathbf{g}(t-k\theta)]. \tag{7.11}$$

If g is continuous and bounded, we find the following characteristic formula for $Q_{1,2}(t)[g]$

$$\begin{aligned}
Q_{1,2}(\tau)[g](x,v) &= \int_0^\tau d\tau_2 S(\tau-\tau_2)\mathcal{L}S(\tau_2)[g](x,v)\\
&= \int_0^\tau d\tau_2\int_{\mathbb{S}^2\times\mathbb{R}^3}\Big(S(\tau_2)[g](x-(t-\tau_2)v,v') + S(\tau_2)[g](x-(t-\tau_2)v,\bar{v}_2')\\
&\quad - S(\tau_2)[g](x-(t-\tau_2)v,\bar{v}) - S(\tau_2)[g](x-(t-\tau_2)v,v)\Big)\big((v-\bar{v})\cdot\eta\big)_+ M(v_*)d\eta d\bar{v}d\tau_2\\
&= \int_{\mathbb{G}^{(2),0}_{(1,1)}} g_2(\xi_2^0)d\Lambda^{(2)}_{(1,1)} - \int_{\mathbb{G}^{(2),0}_{(1,-1)}} g_2(\xi_2^0)d\Lambda^{(2)}_{(1,-1)}
\end{aligned}$$

where we denote as in the previous paragraph

$$g_n(Z_n) := \sum_{i=1}^n g(z_i).$$

We can iterate this construction:

$$Q_{\underline{n}}(t)[g](z_1) = \sum_{(\tilde{a}_i,\tilde{s}_i)_i}\prod_{i=1}^{n_K-1}\tilde{s}_i\int_{\mathbb{G}^{\underline{n},0}_{(\tilde{a}_i,\tilde{s}_i)_i}(z_1)} g_{n_K}(\xi^0_{n_K})\ d\Lambda^{\underline{n}}_{(\tilde{a}_i,\tilde{s}_i)_i}. \tag{7.12}$$

The first term of (7.11) corresponds to the main part of $\mathbb{E}_\varepsilon\big[\zeta_\varepsilon^t(h)\zeta_\varepsilon^0(g)\big]$. The second one is treated by the following L^2 estimation:

Proposition 7.5. *There exists a constant C such that for any $g\in L^2(M(v)dz)$, and $\underline{n} := (n_1,\cdots,n_k)$,*

$$\big\|Q_{\underline{n}}(k\theta)g\big\|_{L^2(M^2(v)dz)} \leq \big(C(k-1)\theta\big)^{\frac{n_{k-1}}{2}}\big(C\theta\big)^{\frac{n_k-n_{k-1}}{2}}\|g\|_{L^2(M(v)dz)}. \tag{7.13}$$

Proof. The proof is given in section 4.4 of [4]. We suppose that g is continuous in order to use the pseudocharacteristic formula, and we conclude by density.

Using Cauchy-Schwartz inequality,

$$\begin{aligned}
&\left\|Q_{\underline{n}}(k\theta)g\right\|^2_{L^2(M^2(v)dz)}\\
&= \int_{\mathbb{D}} \left(\sum_{(\tilde a_i,\tilde s_i)_i} \prod_{i=1}^{n_K-1} \tilde s_i \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} g_{n_K}(\xi^0_{n_K})\; d\Lambda^{\underline{n}}_{(\tilde a_i,\tilde s_i)_i} \right)^2 M^2(v_1) dz_1\\
&\le \int_{\mathbb{D}} \left(M(z_1) \sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} d\Lambda^{q,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} \right) \sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} g^2_{n_K}(\xi^0_{n_K})\; d\Lambda^{b,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} M(v_1) dz_1
\end{aligned}$$

where

$$d\Lambda^{b,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} := M(v_1)dz_1 \prod_{i=2}^{n_K} \frac{\big((v^\varepsilon_{a(i)}(t_i^+) - \bar v_i)\cdot \eta_i\big)_+}{1+\left\|v^\varepsilon_{a(i)}(t_i^+) - \bar v_i\right\|} M(\bar v_i) d\bar v_i d\eta_i dt_i,$$

$$d\Lambda^{q,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} := M(v_1)dz_1 \prod_{i=2}^{n_K} \big((v^\varepsilon_{a(i)}(t_i^+) - \bar v_i)\cdot \eta_i\big)_+ \Big(1+\left\|v^\varepsilon_{a(i)}(t_i^+) - \bar v_i\right\|\Big) M(\bar v_i) d\bar v_i d\eta_i dt_i.$$

From (7.2) we have the bound

$$\left(M(z_1) \sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} d\Lambda^{q,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} \right) \le \big(C(k-1)\theta\big)^{n_{k-1}} \big(C\theta\big)^{n_k - n_{k-1}}.$$

On the other hand, using the representation formula in the reverse sense,

$$\begin{aligned}
&\sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} g^2_{n_K}(\xi^0_{n_K})\; d\Lambda^{b,\underline{n}}_{(\tilde a_i,\tilde s_i)_i}\\
&\le n_K \sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} \big(g^2\big)_{n_K}(\xi^0_{n_K})\; d\Lambda^{b,\underline{n}}_{(\tilde a_i,\tilde s_i)_i}\\
&\le n_K \int_\theta^{k\theta} dt_2 \cdots \int_\theta^{t_{n_{k-1}-1}} dt_{n_K} \int_0^\theta dt_{n_{k-1}+1} \cdots \int_0^{t_{n_k-1}} dt_{n_k} S(t-t_2)|L^b|\cdots|L^b|S(t_n)g^2
\end{aligned}$$

with

$$|L^b|g(v) := \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big(g(v') + g(\bar v') + g(v) + g(\bar v)\big) \frac{((v-\bar v)\cdot\eta)_+}{1+\|v-\bar v\|} M(\bar v) d\eta\, d\bar v$$

and

$$(g^2)_{n_K}(Z_{n_K}) := \sum_{i=1}^{n_k} g^2(z_i).$$

Lemma 7.6. *The operator $|L^b| : L^1(M(v)dz) \to L^1(M(v)dz)$ is bounded.*

Proof. For $f \in L^1(M(v)dz)$, using the change of variables $(v,\bar v,\eta) \mapsto (v',\bar v',\eta)$ sending $(v-\bar v)\cdot\eta)_+ dv\, d\bar v\, d\eta \to (v'-\bar v')\cdot\eta)_- dv'\, d\bar v'\, d\eta$,

$$\int_{\mathbb{D}} |L^b| f(z) M(v) dz = 4 \int \int_{\mathbb{D}\times\mathbb{S}^2\times\mathbb{R}^3} f(z) M(v) M(\bar v) dz d\eta d\bar v \le 16\pi \|f\|_{L^1(M(v)dz)}.$$

$\square$

We use now that $S(t)$ conserves the $L^1(M(v)dz)$ norm. Integrating the times variables,

$$\int_{\mathbb{D}} \sum_{(\tilde a_i,\tilde s_i)_i} \int_{\mathbb{G}^{\underline{n},0}_{(\tilde a_i,\tilde s_i)_i}(z_1)} g^2_{n_K}(\xi^0_{n_K})\, d\Lambda^{b,\underline{n}}_{(\tilde a_i,\tilde s_i)_i} M(v_1)dz_1 \leq \frac{(C(k-1)\theta)^{n_{k-1}}(C\theta)^{n_k-n_{k-1}}}{n_{k-1}!(n_k-n_{k-1})!}\|g\|_{L^2(M(v)dz)}$$

which concludes the proof of the proposition. □

Because $\|\mathbf{g}(t)\|_{L^2(M(z)dz)}$ is decreasing, we have for $\|h\|<\infty$ (we use here the weight of the norm $\|h\| \approx \sup\left|M^{-1}g\right|$).

$$\left|\left\langle h, \sum_{k=1}^K \sum_{\substack{(n_j)_{j\leq k-1}\\ n_j\leq 2^j}} \sum_{n_k>2^k} Q_{\underline{n}}(k\theta)\mathbf{g}(t-k\theta)\right\rangle_{L^2(M(v)dz)}\right| \leq \sum_{k=1}^K (C^2t\theta)^{k/2}\|h\|\|g\|_{L^2(M(v)dz)} \leq Ct^{1/2}\theta^{1/2}\|h\|\|g\|. \tag{7.14}$$

Using all the estimations (4.1), (5.1), (6.1), (7.10) and (7.14), we finally get that

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] = \left\langle h,\, \mathbf{g}(t)\right\rangle_{L^2(M(v)dz)} + O\Big(\Big(Ct\theta^{1/2} + (Ct)^{2^{t/\theta}}\varepsilon^{\alpha/2}\Big)\|h\|\big(\|g\|+\|\nabla g\|\big)\Big). \tag{7.15}$$

This concludes the proof of the main theorem.

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Author Declarations

The author has no conflicts to disclose.

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