

# BOLTZMANN-GRAD LIMIT OF A HARD SPHERE SYSTEM IN A BOX WITH ISOTROPIC BOUNDARY CONDITIONS

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ABSTRACT. In this paper we present a rigorous derivation of the Boltzmann equation in a compact domain with isotropic boundary conditions. We consider a system of  $N$  hard spheres of diameter  $\epsilon$  in a box  $\Lambda := [0, 1] \times (\mathbb{R}/\mathbb{Z})^2$ . When a particle meets the boundary of the domain, it is instantaneously reinjected into the box with a random direction, but conserving kinetic energy. We prove that the first marginal of the process converges in the scaling  $N\epsilon^2 = 1$ ,  $\epsilon \rightarrow 0$  to the solution of the Boltzmann equation, with the same short time restriction of Lanford's classical theorem.

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## 1. INTRODUCTION

A simple model of gas is the hard sphere gas. Each molecule is described as a little sphere of diameter  $\epsilon > 0$  moving freely along straight lines in a domain  $\Lambda \subset \mathbb{R}^d$ ,  $d \geq 2$ , and interacting with the other molecules only at distance  $\epsilon$ . A classical problem is the study of limits of such a system

when the number of particles  $N$  goes to infinity. Of course this limit depends deeply on the relation between  $N$  and  $\epsilon$ . We shall focus here on an intermediate limit which bridges the microscopic scale proportional to  $\epsilon$ , and the macroscopic scales where we only see average quantities as the temperature or the mean velocity.

At a mesoscopic scale we look at the distribution  $f(t, x, v)$  of one particle in the phase space  $\Lambda_x \times \mathbb{R}_v^d$  at time  $t \geq 0$ . In the case of hard spheres, the only interesting limit of this type is the Boltzmann-Grad scaling  $N\epsilon^{d-1} = 1$  (see [17]), for which the density of the gas goes to 0 as  $\epsilon$ , but the mean distance traveled by a particle between two collisions (the mean free path) is constant. In this regime we expect that the coordinates of two randomly chosen particles are "almost independent" and that the limiting one-particle distribution is governed by the Boltzmann equation

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= Q(f, f), \\ Q(f, f)(v) &:= \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left( f(v - \nu \cdot (v - v_*)\nu) f(v_* + \nu \cdot (v - v_*)\nu) \right. \\ &\quad \left. - f(v) f(v_*) \right) b(v - v_*, \nu) dv_* d\nu \end{aligned}$$

where the operator  $Q(f, f)(x, v)$  describes the variation due to collisions and  $b(v - v_*, \nu)$  is a given collision kernel ([3]).

In a fundamental paper [21], Lanford stated a convergence theorem of the hard spheres system for a short time  $t < T^*$ ,  $T^*$  depending on the initial condition. A detailed proof has been provided later on, see in particular [14, 24] for quantitative bounds on the convergence error (including smooth potentials with finite range) in the case of a domain without boundary (see also [26, 7, 8, 15]). There also exists a proof of long time convergence when the domain is  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and the gas is very diluted (meaning in particular that the initial distribution of particles  $f_0(x, v)$  is bounded by a Gaussian  $\eta e^{-\frac{|x|^2 + |v|^2}{2}}$ , with  $\eta > 0$  sufficiently small) (see [20]).

Adding boundaries is important especially in relation with the problem of nontrivial stationary solutions, which is one of the main domains of application of the Boltzmann equation (see [6, 12, 4]). Even without considering the stationary problem, the presence of boundaries leads to several delicate issues. A first one is the modeling itself; a problem which goes back to the origins of kinetic theory (see [6] for a discussion on several different kinds of reflection law).

Let  $\Lambda$  be an open domain of dimension  $d$  with a smooth boundary. A first example of reflection law is the specular reflection: when a particle hits the boundary at point  $x$  with velocity  $v$ , it is reflected with velocity  $v' := v - 2(n(x) \cdot v)n(x)$  where  $n(x)$  is the inner normal vector of  $\partial\Lambda$  in  $x$ . Thus at the boundary the distribution verifies the condition  $f(t, x, v) = f(t, x, v')$ . This dynamics encodes complications because of possible focusing. Note however Théophile Dolmaire's thesis (see [9, 10]) where a Lanford theorem in the half space has been proved.

A second and very famous model are the thermal (or Gaussian, diffusive) boundary conditions: when a particle meets the boundary at point  $x$ , it is reflected with velocity  $v'$  following the probability law

$$d\mathbb{P}(v') = (v' \cdot n(x))_+ M_w(x, v') dv', \quad M_w(x, v) := \frac{e^{-\frac{|v|^2}{2T(x)}}}{(2\pi)^{\frac{d-1}{2}} T(x)^{\frac{d+1}{2}}}$$

where  $T(x)$  represents the temperature of the boundary at point  $x$ . Thus the distribution verifies for all  $(x, v)$  such that  $x \in \partial\Lambda$ ,  $v \cdot n(x) > 0$

$$f(t, x, v) = M_w(x, v) \int f(t, x, u) (u \cdot n(x))_- du.$$

The system of hard spheres with these boundary conditions has fluctuating energy, at variance with the Lanford's setting which typically models an isolated system.

The usual argument for the rigorous convergence in the Boltzmann-Grad limit looks hardly adaptable in this case. Note in particular, that we should answer first the following question (which can be found in [6]): does the Boltzmann equation admit stationary solutions when the system is in contact with a reservoir of prescribed temperature? The question is not answered in full generality. However if the temperature at the boundary is smooth with small variations, then there are existence theorems for the stationary Boltzmann equation for any Knudsen number (the inverse of the mean free path) (see [18, 19, 11, 13, 12]). Unfortunately the scheme of Lanford's proof requires a priori estimates involving infinitely many reference distributions with increasing temperature (see for example chapter 5 of [14] or Section 4.2 of the present paper). In addition note that we do not even know if the hard sphere dynamics with Gaussian reflection is a well defined process (see [5]; however this looks just a technical problem and the process is indeed well defined for suitable smooth interactions (see [16])).

In the following we will therefore investigate a simpler model: isotropic reflection model. It has been described by Levermore in [23]. In this model, a particle conserves its velocity when it reaches the boundary and is reflected in a random direction. More precisely, an incoming particle with coordinates  $(x, v)$  has outgoing velocity  $v'$  following the law

$$d\mathbb{P}(v') = c_d \frac{(v' \cdot n(x))_+}{|v'|^d} \delta_{|v|=|v'|} dv', \quad \text{where } c_d := \left( \int_{\mathbb{S}^{d-1}} (\omega \cdot \bar{e}_1)_+ d\omega \right)^{-1}$$

with  $\delta$  the Dirac mass,  $c_d$  a normalization constant and  $\bar{e}_1$  a unit vector. With this reflection law the distribution has to respect the following boundary condition: for  $x \in \partial\Lambda$ ,  $v \cdot n(x) > 0$ ,

$$f(t, x, v) = \int_{\mathbb{S}^{d-1}} f(t, x, |v|\omega) c_d (n(x) \cdot \omega)_- d\omega.$$

Since the energy is conserved and any Gaussian distribution is a stationary measure, we can expect to be able to adapt Lanford's strategy.

It is worth mentioning that this system in the hydrodynamic limit leads to an adiabatic model: the temperature verifies Neumann's boundary condition  $n(x) \cdot \nabla \theta_{\partial\Omega} = 0$  and the mean velocities verify the Dirichlet boundary condition  $u_{\partial\Omega} = 0$  (as discussed in [22], see also appendix B).

The paper is organized as follows.

We give a proper definition of the process in Section 2 and we derive the evolution law of a symmetric distribution of particles in Section 3: the BBGKY Duhamel series 10. In this part a geometric representation of the Duhamel series is crucial, which is usually referred in the literature as *pseudotrajectories expansion* (see [14]). Section 4 is dedicated to the formal limit, namely the Boltzmann Duhamel series. In Section 5 we state the Lanford's theorem in a domain with stochastic boundary (Theorem 5.1). Section 6 is devoted to the main step of the proof: the "mean" convergence of the hard sphere process to its formal limit (the *punctual process*).

In Section 6 we shall need detailed considerations involving the hard-sphere process. In order to simplify this step we have to restrict to the simple domain  $\Lambda := [0, 1] \times (\mathbb{R}/\mathbb{Z})^2$ . For simplicity of presentation we will work on this domain throughout the whole paper, although we will really use this simplification in Section 6 only. One of the main ingredients is that outside a small set of particle configurations and of time variables, hard sphere and punctual process have the same velocities. Then the error between hard spheres and punctual particles comes only from shifts of size  $\epsilon$  at each collision. If we look at a general domain  $\Lambda \subset \mathbb{R}^3$  we lose such simple feature. We believe that the theorem remains true, but the proof would be certainly more delicate.

## 2. THE MODEL

We will now give a precise definition of the process.

We will use the notation  $\mathbf{a}_n = (a_1, \dots, a_n)$ .

We want to describe the motion of  $N$  hard spheres of diameter  $\epsilon$  in a smooth domain  $\Lambda$ . In the following we take  $\Lambda = [0, 1] \times \mathbb{T}^2$  where  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ . It is a simple model of compact domain with boundary: the boundary is separated in two parallel planes and a particle has to cross a minimum distance between two reflections. We use this simplification only in section 6. More general domains could be considered at the expense of more intricate geometrical estimates.

The particles move along straight lines until they meet either the boundary of  $\Lambda$  or another particle. When two particles meet, there is an elastic *collision*. When a particle reaches the boundary of  $\Lambda$  at the point  $x$  with incoming velocity  $v^{\text{in}}$ , it is *reflected* at the same point  $x$  with velocity  $v^{\text{out}}$  following a probability law  $K_x(v^{\text{in}}|v^{\text{out}})(v^{\text{out}} \cdot n(x))_+ dv^{\text{out}}$ , where  $n(x)$  is the inner normal vector of the surface  $\partial\Lambda$ . We say that there is a *reflection*.

It is not obvious that such process is well defined, and we will restrict in the following to the case of isotropic reflection :

$$(1) \quad K_x(v^{\text{in}}|v^{\text{out}})(v^{\text{out}} \cdot n(x))_+ dv^{\text{out}} = \frac{c_3 \delta(|v^{\text{in}}| - |v^{\text{out}}|)(v^{\text{out}} \cdot n(x))_+ dv^{\text{out}}}{|v^{\text{out}}|^3},$$

$$\text{with } c_3 := \left( \int_{\mathbb{S}^2} (\omega \cdot \vec{\mathbf{e}}_1)_+ d\omega \right)^{-1} = \frac{1}{\pi}$$

where  $\vec{\mathbf{e}}_1$  is the vector  $(1, 0, 0) \in \mathbb{R}^3$ .

Note that in the case of isotropic reflection both measures  $K_x(v^{\text{in}}|v^{\text{out}})(v^{\text{out}} \cdot n(x))_+ dv^{\text{out}}$  and  $K_x(v^{\text{in}}|v^{\text{out}})(v^{\text{in}} \cdot n(x))_- dv^{\text{in}}$  are a probability measure. Thus we can in a certain sense reverse the hard sphere process. This will be the main ingredient of the construction of the process below. In the case of Gaussian boundary conditions,  $K_x(v^{\text{in}}|v^{\text{out}})(v^{\text{in}} \cdot n(x))_- dv^{\text{in}} = M_w(x, v^{\text{out}})(v^{\text{in}} \cdot n(x))_- dv^{\text{in}}$  which is not a probability measure and the following strategy cannot be applied.

**2.1. Construction of the stochastic process.** To discuss the well-posedness of the system, we introduce the phase space

$$(2) \quad \mathcal{D}_\epsilon^N := \{(\mathbf{x}_N, \mathbf{v}_N) = (x_1, \dots, x_N, v_1, \dots, v_N) \in \Lambda^N \times \mathbb{R}^{3N} \mid \forall i \neq j, |x_i - x_j| > \epsilon\}$$

and the probability space

$$(3) \quad \Omega := \{\bar{\omega} = (\omega^j)_{j \in \mathbb{Z}^*} \text{ with } \omega^j \in \mathbb{S}^2, \omega^j \cdot \vec{\mathbf{e}}_1 > 0 \text{ if } j > 0, \omega^j \cdot \vec{\mathbf{e}}_1 < 0 \text{ if } j < 0\}.$$

We assign to  $\Omega$  the measure  $d\mathbb{P}(\bar{\omega} = (\omega^j))$  which is the probability measure of sequences of independent random variables such that  $\omega^j$  follows the law  $c_3(\omega^j \cdot \vec{\mathbf{e}}_1)_- d\omega^j$  for  $j < 0$  and  $c_3(\omega^j \cdot \vec{\mathbf{e}}_1)_+ d\omega^j$  for  $j > 0$ .

Let  $\Gamma$  be the function on  $\partial\Lambda$  such that

$$\Gamma(x) := \begin{cases} 1 & \text{for } x \in \{0\} \times \mathbb{T}^2 \\ -1 & \text{for } x \in \{1\} \times \mathbb{T}^2. \end{cases}$$

We define now a dynamics on the extended phase space  $\mathcal{D}_\epsilon^N \times \Omega^N$ . Let

$$(\mathbf{z}_N^0, \bar{\omega}_N^0) = (\mathbf{x}_N^0, \mathbf{v}_N^0, \bar{\omega}_1^0, \dots, \bar{\omega}_N^0) \in \mathcal{D}_\epsilon^N \times \Omega^N$$

and  $t > 0$  be a time,

- $(\mathbf{z}_N^\epsilon(0), \bar{\omega}_N^\epsilon(0)) = (\mathbf{z}_N^0, \bar{\omega}_N^0)$ ;

- until  $z_N^\epsilon(t)$  reaches the boundary of  $\mathcal{D}_\epsilon^N$ ,  $\bar{\omega}_N^\epsilon(t)$  is constant and each  $z_i^\epsilon(t)$  ( $i \in \{1, \dots, N\}$ ) moves along straight lines;
- if  $|x_i^\epsilon(t) - x_j^\epsilon(t)| = \epsilon$ ,  $v_i^\epsilon(t^+)$  and  $v_j^\epsilon(t^+)$  are given by an elastic collision between the two particles :

$$(4) \quad \begin{cases} v_i^\epsilon(t^+) = v_i^\epsilon(t^-) - \nu_{i,j} \cdot (v_i^\epsilon(t^-) - v_j^\epsilon(t^-)) \nu_{i,j} \\ v_j^\epsilon(t^+) = v_j^\epsilon(t^-) + \nu_{i,j} \cdot (v_i^\epsilon(t^-) - v_j^\epsilon(t^-)) \nu_{i,j} \end{cases}$$

with  $\nu_{i,j} := \frac{x_i^\epsilon(t) - x_j^\epsilon(t)}{|x_i^\epsilon(t) - x_j^\epsilon(t)|}$  and where  $t^\pm$  indicate the limit from the future/past.

- if  $x_i^\epsilon(t)$  meets the boundary of  $\Lambda$ , we record the in-coming direction of  $v_i^\epsilon(t^-)$  and  $v_i^\epsilon(t^+)$  takes the out-coming direction :

$$(5) \quad \begin{cases} v_i^\epsilon(t^+) = |v_i^\epsilon(t^-)| \Gamma(x_i^\epsilon(t)) \omega_i^{\epsilon,1}(t^-) \\ \omega_i^{\epsilon,-1} = \Gamma(x_i^\epsilon(t)) \frac{v_i^\epsilon(t^-)}{|v_i^\epsilon(t^-)|} \\ \forall j \in \mathbb{Z}^* \setminus \{-1\}, \omega_i^{\epsilon,j}(t^+) = \omega_i^{\epsilon,j+1}(t^-); \end{cases}$$

then we iterate the process.

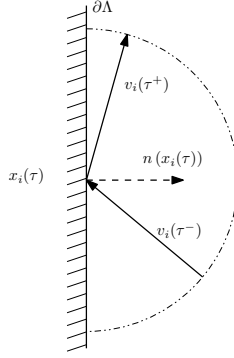


FIGURE 1. Reflection of particle  $i$  at time  $\tau$ .

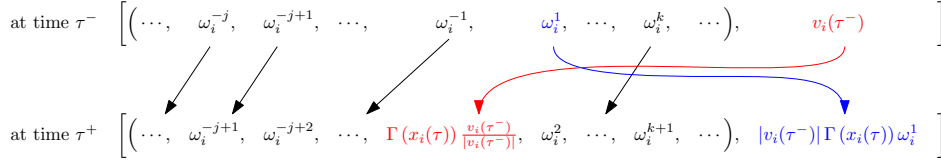


FIGURE 2. Evolution of  $\bar{\omega}_i$  when particle  $i$  has a reflection at time  $\tau$ .

For example in the case with only one particle and after  $k$  reflections,  $\bar{\omega}_1^\epsilon(t)$  is equal to

$$\left( \begin{array}{cccccccc} \dots & \omega_1^{0,-2} & \omega_1^{0,-1} & \Gamma(x_1^1) \frac{v_1^1}{|v_1^1|} & \Gamma(x_1^2) \frac{v_1^2}{|v_1^2|} & \dots & \Gamma(x_1^k) \frac{v_1^k}{|v_1^k|} & \omega_1^{0,k+1} & \dots \\ & -k-2 & -k-1 & -k & -k+1 & & -1 & 1 & \end{array} \right)$$

where  $x_1^i := x_1^\epsilon(t_i)$  and  $v_1^i := v_1^\epsilon(t_i^-)$  are the position of the particle and its incoming velocity at the  $i$ -th reflection.

In the same way we can define the backward dynamics for  $t < 0$ .

Note that the variables  $(\omega_i^j)_{j < 0}$  are used to record the reflections. This will be practical to reconstruct the dynamics backwards, independently of the number of reflections.

**2.2. Well-posedness of the process.** We denote  $\phi_N^{\epsilon,t}(\mathbf{z}_N, \bar{\omega}_N) = (\mathbf{z}_N^\epsilon(t), \bar{\omega}_N^\epsilon(t))$  the flow described above with initial conditions  $(\mathbf{z}_N, \bar{\omega}_N)$ . It is not well defined everywhere and we can have *bad initial data* which lead to

- a collision involving more than two particles at some time,
- two collisions/reflections at the same time,
- infinitely many collisions/reflections during a finite time,
- grazing collisions/reflections.

However such "pathological" trajectories are exceptional: if we denote  $\mathcal{B}^N \subset \mathcal{D}_\epsilon^N \times \Omega^N$  the set of bad initial data and  $\mathbb{P}^N(d\bar{\omega}_N) := \mathbb{P}(d\bar{\omega}_1) \otimes \cdots \otimes \mathbb{P}(d\bar{\omega}_N)$ , we have

**Proposition 2.1.** *The pathological set  $\mathcal{B}^N$  is of zero measure for  $d\mathbf{z}_N \mathbb{P}^N(d\bar{\omega}_N)$ , the dynamics on  $\mathcal{D}_\epsilon^N \times \Omega^N$  is well defined for almost all initial data and  $\phi_N^{\epsilon,t}$  conserves the measure: for all Borel sets  $A \subset \mathcal{D}_\epsilon^N \times \Omega^N$ ,  $\phi_N^{\epsilon,t}(A \setminus \mathcal{B}^N)$  is measurable, with the same measure of  $A$ .*

This is an adaptation of the proof of Alexander [1] and it stems from the following lemma. We denotes  $B_R^N$  the euclidean ball of radius  $R$  in  $\mathbb{R}^{3N}$ .

**Lemma 2.2.** *Let  $R > 0$  be given, and let  $\delta$  be a real number in  $(0, \epsilon/2)$ . Let*

$$(6) \quad I := \left\{ (\mathbf{z}_N, \bar{\omega}_N) \in \Lambda^N \times B_R^N \times \Omega^N \mid \begin{array}{l} \text{there are two shocks} \\ \text{(reflections or collisions) during the time interval } [0, \delta] \end{array} \right\}.$$

*Then for  $\epsilon$  small enough,  $|I| \leq C(N, \epsilon, R)\delta^2$ .*

*Proof.* We have to treat the different cases with at least two shocks (reflections or collisions) in the same interval  $[0, \delta]$ .

First consider the case where at least two particles are involved. Because velocities are bounded by  $R$ , if particle  $i$  has a collision with particle  $j$  in the time interval  $[0, \delta]$ , the distance  $d(x_i(0), x_j(0))$  is bounded by  $2\delta R$ . In the same way particle  $i$  can have a reflection only if  $d(x_i(0), \partial\Lambda) < \delta R$ . Thus there can be two shocks only if  $(\mathbf{z}_N(0), \bar{\omega}_N(0))$  is in

$$I_1 := \left\{ (\mathbf{z}_N, \bar{\omega}_N) \in \Lambda^N \times B_R^N \times \Omega^N \mid \begin{array}{l} \exists i \neq j, \text{ and } k, l \notin \{i, j\}, \\ \left( \begin{array}{l} d(x_i, B_\epsilon^1(x_k) \cup \partial\Lambda) \leq 2\delta R \\ d(x_j, B_\epsilon^1(x_l) \cup \partial\Lambda) \leq 2\delta R \end{array} \right) \end{array} \right\}$$

where  $d(x, A)$  is the euclidean distance of a point  $x$  to a subset  $A$ . The set  $I_1$  is of measure at most  $[2R\delta|\partial\Lambda| + (4/3)\pi((\epsilon + 2R\delta)^3 - \epsilon^3)]^2 \leq C(N, \epsilon, R)\delta^2$  for  $\delta$  small enough.

Then we pass to the case where there is one particle which reflects twice the boundary on the interval  $[0, \delta]$ . This case is made impossible for  $\delta < 1/R$  because between two reflections a particle has to cross  $(0, 1) \times \mathbb{T}^2$ .  $\square$

*Proof of Proposition 2.1.* Fix  $R > 0$  and  $t > 0$ . Let  $\delta < \epsilon/2$  be a small parameter such that  $t/\delta$  is an integer. Lemma 2.2 shows that there exists a subset  $I_0(\delta, R)$  of  $\Lambda^N \times B_R^N \times \Omega^N$  such that outside of  $I_0$  there is at most one shock in the time interval  $[0, \delta]$ . Its measure is at most  $C(N, \epsilon, R)\delta^2$ . Observe that the set leading to grazing shocks is of zero measure.

Note that the flow is conservative where it is well defined. Indeed the boundary conditions for collisions are conservative. For the reflection, the map

$$(x, v, \omega^1) \mapsto \left( x + \tau(x, v)v + (t - \tau(x, v))|v|\omega^1, |v|\omega^1, \omega^{-1}(t) := \Gamma(x + \tau(x, v)v)/|v| \right)$$

where  $\tau(x, v)$  is the time of travel of one particle to the boundary, is conservative. For the other  $\omega^i, i \notin \{0, -1\}$ , we just apply the shift  $(\omega^i) \mapsto (\omega^{i+1})$ , which is also conservative.

Hence there is no pathological trajectory in  $\Lambda^N \times B_R^N \times \Omega^N \setminus I_0(\delta, R)$ . We recall that the set  $\Lambda^N \times B_R^N \times \Omega^N$  is stable under the flow (the energy  $\frac{1}{2} \sum_i |v_i|^2$  is conserved). We iterate the procedure and construct a set  $I_1(\delta, R)$  such that outside  $I_1$  there is at most one shock in the interval  $[\delta, 2\delta]$ . Because the flow is conservative,  $I_1(\delta, R)$  is of size at most  $C(N, \epsilon, R)\delta^2$ . More generally we can construct a sequence of sets  $(I_k(\delta, R))_k$  such that outside  $\bigcup_{0 \leq k \leq K} I_k(\delta, R)$  there is no pathological trajectory during the interval  $[0, (K+1)\delta]$ .

We define  $I(\delta, t, R)$  as:

$$I(\delta, t, R) := \bigcup_{0 \leq k \leq t/\delta} I_k(\delta, R).$$

Each set  $I(\delta, t, R)$  is of size at most  $C(N, \epsilon, R)\delta^2 \cdot t/\delta = C(N, \epsilon, R)t\delta$ . Setting

$$I(t, R) := \bigcap_{n \in \mathbb{N}^*} I(t/n, t, R),$$

$I(t, R)$  is of null-measure and outside it there is no pathological trajectory on  $[0, t]$ . We take the union  $\mathcal{B}^N$  of the  $I(t, R)$  for a sequence of  $t$  and  $R$  going to infinity. Outside it there is no pathological trajectory. This concludes the proof.  $\square$

### 3. THE BBGKY HIERARCHY AND ITS PSEUDO-TRAJECTORIES

**3.1. Stochastic semigroup and expression of the hierarchy.** We will use the following notation: for  $1 \leq k < l \leq n$  two integers,  $\mathbf{a}_{k,l} = (a_k, \dots, a_l)$ .

We want to study a system of  $N$  identical particles, distributed at time zero according to a probability  $\mu_N^0$  on  $\mathcal{D}_\epsilon^N$ . Because all the particles are indiscernible, the measure  $\mu_N$  is assumed stable under permutation of particle labels.

We define by duality the semi-group  $T_N^\epsilon(t)$  on  $\mathcal{M}_0(\mathcal{D}_\epsilon^N)$ , the space of finite signed measures  $\mu_N$  such that the set of bad trajectories  $\mathcal{B}^N \subset \mathcal{D}_\epsilon^N \times \Omega^N$  is of measure zero for  $\mu_N \otimes \mathbb{P}^N$ . For any bounded continuous function  $\varphi : \mathcal{D}_\epsilon^N \rightarrow \mathbb{R}$ ,

$$(7) \quad \int_{\mathcal{D}_\epsilon^N} \varphi(\mathbf{z}_N) (T_N^\epsilon(t)\mu_N)(d\mathbf{z}_N) := \int_{\mathcal{D}_\epsilon^N} \left( \int_{\Omega^N} \varphi[\phi_N^{\epsilon,t}(\mathbf{z}_N^0, \bar{\omega}_N^0)] d\mathbb{P}^N(\bar{\omega}_N^0) \right) \mu_N(d\mathbf{z}_N^0).$$

In the case of density measures, we have an explicit formula :

**Proposition 3.1.** For  $W_{0,N} \in L^1 \cap L^\infty(\mathcal{D}_\epsilon^N)$ ,

$$(8) \quad T_N^\epsilon(t)(W_{0,N}(\mathbf{z}_N)d\mathbf{z}_N) = \left( \int_{\Omega^N} W_{0,N}[\phi_N^{\epsilon,-t}(\mathbf{z}_N^0, \bar{\omega}_N^0)] d\mathbb{P}^N(\bar{\omega}_N^0) \right) d\mathbf{z}_N^0, \text{ with } t > 0.$$

*Proof.* It is a direct application of the conservation of measure of  $\phi_N^{\epsilon, t}$ , Proposition 2.1.  $\square$

We denote  $W_{0,N}$  the initial density distribution of particles at time 0 and  $W_N(t)$  the evolution of this distribution. To observe some limit behavior, we have to fix the number of particles  $s$  that we study. Let  $(f_N^{(s)})_{1 \leq s \leq N}$  be the marginals of  $W_{0,N}$  and  $(f_{0,N}^{(s)}(t))_{1 \leq s \leq N}$  be the marginals of  $W_N(t)$ :

$$(9) \quad \forall \mathbf{z}_s \in \mathcal{D}_\epsilon^s, f_{0,N}^{(s)}(\mathbf{z}_s) := \int W_{0,N}(\mathbf{z}_s, \mathbf{z}_{s+1,N}) d\mathbf{z}_{s+1,N}$$

and the same thing for  $f_N^{(s)}(t)$ . By convention we extend functions on  $\mathcal{D}_\epsilon^s$  by 0 outside  $\mathcal{D}_\epsilon^s$ .

The following theorem describes the evolution of the marginals:

**Theorem 3.2.** *Let  $W_{0,N}$  be a function in  $L^\infty \cap L^1(\mathcal{D}_\epsilon^N)$ . Its marginals  $(f_{0,N}^{(s)})_s$  verify the series expansion of the BBGKY hierarchy (we will call it in the following the BBGKY hierarchy):*

$$(10) \quad \begin{aligned} f_N^{(s)}(t) &= \sum_{r=0}^{N-s} \alpha(N-s, r) \epsilon^{2r} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{r-1}} dt_1 \cdots dt_r T_s^\epsilon(t-t_1) C_{s,s+1}^\epsilon \cdots \\ &\cdots C_{s+r-1, s+r}^\epsilon T_{s+r}^\epsilon(t_r) f_{0,N}^{(s+r)} =: \sum_{r=0}^{N-s} \alpha(N-r, N) \epsilon^{2r} Q_{s,s+r}^\epsilon(t) f_{0,N}^{(s+r)} \end{aligned}$$

where  $\alpha(r, s) = r(r-1) \cdots (r-s+1)$  and  $C_{s,s+1}^\epsilon$  is the collision operator:

$$(11) \quad C_{s,s+1}^\epsilon := \sum_{i=1}^s C_{s,s+1}^{\epsilon, i, +} - C_{s,s+1}^{\epsilon, i, -}$$

$$(12) \quad \begin{aligned} &C_{s,s+1}^{\epsilon, i, +} f^{(s+1)}(\mathbf{z}_s) \\ &:= \int_{\mathbb{S}^2 \times \mathbb{R}^3} f^{(s+1)}(\cdots, x_i, v'_i, \cdots, x_i + \epsilon v, v'_{s+1}) (\nu \cdot (v_{s+1} - v_i))_+ dv dv_{s+1} \end{aligned}$$

$$(13) \quad C_{s,s+1}^{\epsilon, i, -} f^{(s+1)}(\mathbf{z}_s) := \int_{\mathbb{S}^2 \times \mathbb{R}^3} f^{(s+1)}(\mathbf{z}_s, x_i + \epsilon v, v_{s+1}) (\nu \cdot (v_{s+1} - v_i))_- dv dv_{s+1}$$

and  $(v_i, v_{s+1})$  is the scattering of  $(v'_i, v'_{s+1})$  (see Equation 4).

The strategy of the proof (presented in section 3.3 below) is an adaptation of [25]. We begin by expanding the  $Q_{s,s+r}^\epsilon(t) f_{0,N}^{(s+r)}$ . We define  $\mathfrak{A}_s^{s+r}$  as the set of *collision trees*

$$(14) \quad \mathfrak{A}_s^{s+r} := \{a := (a(k))_{s < k \leq k+r} \in \mathbb{N}^r, 1 \leq a(k) \leq k-1\}$$

and  $\sigma := (\sigma_k)_{s < k \leq s+r} \in \{\pm 1\}^r$ . Then we have

$$(15) \quad \begin{aligned} Q_{s,s+r}^\epsilon(t) f_{0,N}^{(s+r)} &= \sum_{(a, \sigma) \in \mathfrak{A}_s^{s+r} \times \{\pm 1\}^r} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{r-1}} dt_r \cdots dt_1 \\ &T_s^\epsilon(t-t_1) C_{s,s+1}^{\epsilon, a(s+1), \sigma_{s+1}} \cdots C_{s+r-1, s+r}^{\epsilon, a(s+r), \sigma_{s+r}} T_{s+r}^\epsilon(t_r) f_{0,N}^{(s+r)}. \end{aligned}$$

Because  $T_s^\epsilon$  is the semigroup associated to the hard sphere process, we can rewrite the integral with a characteristic formula. It is the objective of the next section.



**3.2. The pseudotrajectories development.** We begin by rewriting 10 with an explicit "characteristic" formula associated to the *interacting process*. It gives a precise sense to solution of the BBGKY hierarchy with no regularity assumed on initial data, as measures. Ultimately it will permit to compare the hierarchy with its formal limit.

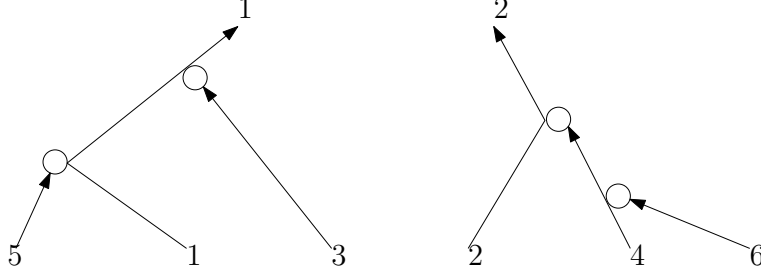


FIGURE 3. Pseudotrajectory associated with collision parameters  $((1, -)_3, (2, +)_4, (1, +)_5, (4, -)_6)$ .

We define for  $s, r \in \mathbb{N}$  the pseudotrajectory  $\zeta_{s, s+r}^\epsilon(\tau, t, \mathbf{z}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{r+s}, a, \boldsymbol{\sigma})$  where :

- $(a, \boldsymbol{\sigma}) \in \mathfrak{A}_s^{s+r} \times \{\pm 1\}^r$ ,
- $\tau \in [0, t]$ ,
- $(\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r} := (t_i, \nu_i, \bar{v}_i)_{s < i \leq s+r} \in (\mathbb{R} \times \mathbb{S}^2 \times \mathbb{R}^3)^r$  the *collisions parameters*, with  $t > t_{s+1} > \dots > t_{s+r} > 0$ ,
- $\bar{\boldsymbol{\omega}}_{r+s} = (\bar{\omega}_1, \dots, \bar{\omega}_{r+s}) \in \Omega^{r+s}$  the *reflections parameters* of all particles.

The number of particles of  $\zeta_{s, s+r}^\epsilon(\tau)$  is not constant: for  $\tau$  between  $t_k$  and  $t_{k+1}$  there are  $s+k$  particles (by convention,  $t_s := t$  and  $t_{r+s+1} = 0$ ). Finally we denote  $\bar{\boldsymbol{\omega}}_{r+s}^\epsilon(\tau)$  the evolution of the reflection parameters. We define  $(\zeta_{s, s+r}^\epsilon(\tau), \bar{\boldsymbol{\omega}}_{r+s}^\epsilon(\tau))$  by

- $(\zeta_{s, s+r}^\epsilon, \bar{\boldsymbol{\omega}}_{r+s}^\epsilon)(t) := (\mathbf{z}_s, \bar{\boldsymbol{\omega}}_{s+r})$ ,
- for  $\tau \in (t_{k+1}, t_k)$ ,  $\bar{\boldsymbol{\omega}}_{k+1, s+r}^\epsilon(\tau)$  is constant and  $(\zeta_{s, s+r}^\epsilon(\tau), \bar{\boldsymbol{\omega}}_{1, k}(\tau))$  follow the hard sphere backward flow described at section 2.1,
- at time  $t_k^+$ , a particle is added at position  $x_k^\epsilon(t_k) = x_{a(k)}^\epsilon(t_k) + \epsilon \nu_k$  and velocity  $\bar{v}_k$ . If  $\sigma = 1$ , we have  $(\bar{v}_k - v_{a(k)}(t_k^+)) \cdot \nu_k > 0$  and we apply the scattering of velocities. Else if  $\sigma_k = -1$ , we do not apply scattering.

Here and below, with a slight abuse of notation,  $(x^\epsilon(\tau), v^\epsilon(\tau))$  designate the coordinates of pseudotrajectories (which are different from coordinates of the stochastic trajectories introduced in section 2.1).

We denote  $\mathcal{G}^\epsilon(\mathbf{z}_s, t, a, \boldsymbol{\sigma})$  the set of admissible coordinates, *i.e.* the  $((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{r+s})$  such that the pseudo-trajectory is well defined according to the previous iteration, and

$$d\Lambda_{a, \boldsymbol{\sigma}}^\epsilon((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) = \left( \prod_{k=s+1}^{s+r} \left[ \sigma_k \nu_k \cdot (\bar{v}_k - v_{a(k)}^\epsilon(t_k^+)) \right]_+ dt_k d\bar{v}_k d\nu_k \right) \mathbb{P}^{s+r}(d\bar{\boldsymbol{\omega}}_{s+r}).$$

Then the formula 10 becomes

$$(16) \quad f_N^{(s)}(t, \mathbf{z}_s) = \sum_{r=0}^{N-s} \alpha(N-s, r) \epsilon^{2r} \sum_{a, \sigma} \sigma_{s+1} \cdots \sigma_{s+r} \\ \times \int_{\mathcal{G}^\epsilon(\mathbf{z}_s, t, a, \sigma)} d\Lambda_{a, \sigma}^\epsilon((t, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) f_{0, N}^{(s+r)}(\zeta^\epsilon(0)).$$

**3.3. Proof of Theorem 3.2.** Now we give a dual form of the previous equation. The idea is to look at the application  $(\mathbf{z}_s, (t, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \mapsto (\zeta^\epsilon(0), \bar{\boldsymbol{\omega}}^\epsilon(0))$  from  $\mathcal{D}_\epsilon^s \times \mathcal{G}^\epsilon(a, \sigma)$  to  $(\mathcal{D}_\epsilon^{s+r} \times \Omega^{s+r}) \setminus \mathcal{B}^{s+r}$  which is a local homeomorphism that sends the measure  $\epsilon^{2r} d\mathbf{z}_s d\Lambda^\epsilon$  into  $d\zeta^\epsilon(0) d\mathbb{P}^{r+s}(\bar{\boldsymbol{\omega}}^\epsilon(0))$ . It is not injective since an initial data can give different pseudotrajectories depending on whether a collision is seen as the *creation* of a particle in the pseudotrajectory or as a *recollision* (a collision between two particles that already exist). Nevertheless the number of collisions is locally constant and finite. Then indexing on  $a, \sigma$  and a new discrete parameter  $M \in [1, \bar{M}] \subset \mathbb{N}$ , we can define the inverse flow  $(\zeta_s^{b, \epsilon}, \bar{\boldsymbol{\omega}}_{s+r}''(t)) (t, \mathbf{z}_{s+r}, \bar{\boldsymbol{\omega}}_{s+r}, a, \sigma, M)$ . Because the number of collisions is almost surely finite,  $\bar{M}$  is also almost surely finite and locally constant (we put  $\bar{M} = 0$  outside the image of the homeomorphism). Thus the equation 10 can be rewritten in the weak sense as for all bounded continuous functions  $\varphi$  defined on  $\mathcal{D}_\epsilon^s$ ,

$$(17) \quad \int_{\mathcal{D}_\epsilon^s} \varphi(\mathbf{z}_s) f_N^{(s)}(t, \mathbf{z}_s) d\mathbf{z}_s = \sum_{r=0}^{N-s} \alpha(N-s, r) \sum_{a \in \mathfrak{A}_s^{s+r}, \sigma} \sigma_{s+1} \cdots \sigma_{s+r} \\ \times \int_{\mathcal{D}_\epsilon^{s+r} \times \Omega^{r+s}} \sum_{M=1}^{\bar{M}} \varphi(\zeta_s^{b, \epsilon}(t)) f_{0, N}^{(s+r)}(\mathbf{z}_{s+r}) d\mathbf{z}_{s+r} d\mathbb{P}^{s+r}(\bar{\boldsymbol{\omega}}_{s+r}).$$

To check this equality, we prove it in the probability space. Note that  $\Omega$  is a compact metric space as countable product of compact spaces, so  $\mathcal{D}_\epsilon^N \times \Omega^N$  is a Polish space.

We find an analogue to  $T_N^\epsilon$  on  $\mathcal{D}_\epsilon^N$ . Let  $\mu_{0, N}$  be a measure in  $\mathcal{M}_0(\mathcal{D}_\epsilon^N \times \Omega^N)$ , the set of finite signed measures stable under permutation of variables and such that  $\mathcal{B}^N$  has zero measure. We define  $H_N(t)\mu_{0, N}$  by duality: for each  $\varphi$  bounded continuous function on  $\mathcal{D}_\epsilon^N \times \Omega^N$ ,

$$(18) \quad \int_{\mathcal{D}_\epsilon^N \times \Omega^N} \varphi(\mathbf{z}_N, \bar{\boldsymbol{\omega}}_N) (H_N(t)\mu_{0, N})(d\mathbf{z}_N, d\bar{\boldsymbol{\omega}}_N) \\ := \int_{\mathcal{D}_\epsilon^N \times \Omega^N} \varphi(\phi_N^{\epsilon, t}(\mathbf{z}_N, \bar{\boldsymbol{\omega}}_N)) \mu_{0, N}(d\mathbf{z}_N, d\bar{\boldsymbol{\omega}}_N).$$

Next, let  $(\mu_{0, N}^{(s)})_{1 \leq s \leq N}$  be the marginals of  $\mu_{0, N}$ , and  $(\mu_N^{(s)}(t))_{1 \leq s \leq N}$  the marginal of  $H_N(t)\mu_{0, N}$ . Then it suffices to prove that for all bounded continuous functions  $\varphi_s$  on  $\mathcal{D}_\epsilon^s \times \Omega^s$ , we have

$$(19) \quad \int_{\mathcal{D}_\epsilon^s \times \Omega^s} \varphi_s(\mathbf{z}_s, \bar{\boldsymbol{\omega}}_s) d\mu_N^{(s)}(t)(\mathbf{z}_s, \bar{\boldsymbol{\omega}}_s) \\ = \sum_{r=0}^{N-s} \alpha(N-s, r) \sum_{a \in \mathfrak{A}_s^{s+r}, \sigma} \sigma_{s+1} \cdots \sigma_{s+r} \int_{\mathcal{D}_\epsilon^{s+r} \times \Omega^{s+r}} \sum_{M=1}^{\bar{M}} \varphi_s(\zeta_s^{b, \epsilon}(t), \bar{\boldsymbol{\omega}}_s''(t)) d\mu_{0, N}^{(s+r)} \\ =: \sum_{r=0}^{N-s} \int_{\mathcal{D}_\epsilon^{r+s} \times \Omega^{r+s}} \varphi_s(\mathbf{z}_s, \bar{\boldsymbol{\omega}}_s) (\mathcal{T}_{s, s+r}(t)\mu_{0, N}^{s+r})(d\mathbf{z}_{s+r}, d\bar{\boldsymbol{\omega}}_{s+r})$$

where the  $\mathcal{T}_{s,s+r}(t)(\mu_{0,N}^{(s+k)})$  are defined by duality.

Because  $\mathcal{D}_\epsilon^N \times \Omega^N$  is a Polish space and following [25], it is sufficient to prove equality 19 for *empirical density distribution* defined as follow.

Let  $(\mathbf{z}_N, \bar{\omega}_N) \in \mathcal{D}_\epsilon^N \times \Omega^N$  be such that the dynamics is well defined. We define the empirical density distribution as the measure

$$\Delta(\mathbf{z}_N, \bar{\omega}_N)(\zeta_1, \bar{\omega}'_1, \dots, \zeta_N, \bar{\omega}'_N) := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N^N} \prod_{i=1}^N \delta_{(\zeta_i, \bar{\omega}'_i) = (z_{\sigma(i)}, \bar{\omega}_{\sigma(i)})}$$

where  $\mathfrak{S}_k^N$  is the set of injection of  $\{1, \dots, k\}$  into  $\{1, \dots, N\}$ .

The  $k$ -th marginal of  $\Delta(\mathbf{z}_N, \bar{\omega}_N)$  is

$$(\Delta(\mathbf{z}_N, \bar{\omega}_N))^{(k)}(\zeta_1, \bar{\omega}'_1, \dots, \zeta_k, \bar{\omega}'_k) = \frac{(N-k)!}{N!} \sum_{\sigma \in \mathfrak{S}_k^N} \prod_{i=1}^k \delta_{(\zeta_i, \bar{\omega}'_i) = (z_{\sigma(i)}, \bar{\omega}_{\sigma(i)})}$$

and we have immediately its evolution with respect to the hard spheres dynamics:

$$H_N(t)\Delta(\mathbf{z}_N, \bar{\omega}_N) = \Delta(\phi_N^{\epsilon, -t}(\mathbf{z}_N, \bar{\omega}_N)).$$

First there exists a finite sequence of times  $0 = t_0 < t_1 < \dots < t_K = t$  such that on each segment  $[t_i, t_{i+1}]$  there is only one collision or a reflection. In addition we impose that if the collision or the reflection implied the particles  $k$  and  $l$ ,

$$\begin{cases} \phi_{N-1}^{\epsilon, \tau}(\mathbf{z}_{1,k-1}^\epsilon(t_i), \mathbf{z}_{k+1,N}^\epsilon(t_i), \bar{\omega}_{1,k-1}^\epsilon(t_i), \bar{\omega}_{k+1,N}^\epsilon(t_i)) \\ \phi_{N-1}^{\epsilon, \tau}(\mathbf{z}_{1,l-1}^\epsilon(t_i), \mathbf{z}_{l+1,N}^\epsilon(t_i), \bar{\omega}_{1,l-1}^\epsilon(t_i), \bar{\omega}_{l+1,N}^\epsilon(t_i)) \end{cases}$$

moves like free flow for  $\tau \in [0, t_{i+1} - t_i]$ . It is possible to construct such sequence because the free flow is well defined and continuous.

Then it is sufficient, in view of the *semigroup property* verified by the marginals, to prove our assumption only for a segment  $[0, t_1]$ .

To simplify the notation, in the following  $(\Delta(\mathbf{z}_N, \bar{\omega}_N))^{(k)}$  will be denoted  $\Delta_k$ . We will prove the formula only for  $\Delta_1$ , the other cases work in the same way (see [25] for more details).

Because there is at most one collision in  $[0, t_1]$ , all the  $\mathcal{T}_{1,1+r}(t)\Delta_{1+r}$  vanish for  $r \geq 2$ .

If there is no collision, then  $\Delta_1(t)$  is just the push-forward of  $\Delta_1$  by the free flow with isotropic reflection, and  $\mathcal{T}_{1,2}(t)\Delta_2$  vanishes. So the formula is verified.

If there is one collision, we can assume without loss of generality that it occurs between particles 1 and 2. Then in  $\mathcal{T}_{1,1}(t)\Delta_1$  the two first particles are replaced by virtual ones :

$$\mathcal{T}_{1,1}(t)\Delta_1 = \frac{1}{N} \left( \delta_{(\tilde{z}_1(t), \bar{\omega}_1)} + \delta_{(\tilde{z}_2(t), \bar{\omega}_2)} + \sum_{i=3}^N \delta_{(z_i(t), \bar{\omega}_i(t))} \right),$$

with  $\tilde{z}_i(t) = (x_i - tv_i, v_i)$ .

In  $\mathcal{T}_{1,2}(t)\Delta_2$ , there are two parts: a first part which corresponds to post collision directions :  $\frac{1}{N} (\delta_{z_1(t), \bar{\omega}_1(t)} + \delta_{z_2(t), \bar{\omega}_2(t)})$ , and a second corresponding to the negative Dirac mass in the virtual particles :  $-\frac{1}{N} (\delta_{(\tilde{z}_1(t), \bar{\omega}_1)} + \delta_{(\tilde{z}_2(t), \bar{\omega}_2)})$ , which compensate the previous error.

Finally we get the expected formula. □

4. BOLTZMANN'S HIERARCHY AND *a priori* ESTIMATES

**4.1. Definition of the Boltzmann hierarchy.** We want now to describe a formal limit of the BBGKY hierarchy 10 when  $\epsilon$  tends to 0 in the scaling  $\epsilon^2 N = 1$ . For  $(f_0^s)_s$  a family of symmetric functions on  $(\Lambda \times \mathbb{R}^3)^s$ , we define the Boltzmann hierarchy and its series expansion (what we call later the *Boltzmann hierarchy*):

$$(20) \quad \begin{aligned} f^s(t) &= \sum_{r=0}^{\infty} \int_0^t \cdots \int_0^{t_{r-1}} dt_1 \cdots dt_r T_s^0(t-t_1) C_{s,s+1}^0 \cdots C_{s+r-1,s+r}^0 T_{s+r}^0(t_r) f_0^{s+r} \\ &=: \sum_{r=0}^{\infty} Q_{s,s+r}^0(t) f_0^{s+r} \end{aligned}$$

where  $T_s^0(t)$  is the semigroup associated to the dynamics of  $s$  punctual particles with reflection in angle (and no collision) and  $C_{s,s+1}^0$  is the formal collision operator for punctual spheres :

$$(21) \quad C_{s,s+1}^0 := C_{s,s+1}^{0,+} - C_{s,s+1}^{0,-}$$

$$(22) \quad C_{s,s+1}^{0,+} f^{s+1}(\mathbf{z}_s) := \sum_{i=1}^s \int_{\mathbb{S}^2 \times \mathbb{R}^3} f^{s+1}(\cdots, x_i, v'_i, \cdots, x_i, v'_{s+1}) (\nu \cdot (v_{s+1} - v_i))_+ dv dv_{s+1}$$

$$(23) \quad C_{s,s+1}^{0,-} f^{s+1}(\mathbf{z}_s) := \sum_{i=1}^s \int_{\mathbb{S}^2 \times \mathbb{R}^3} f^{s+1}(\mathbf{z}_s, x_i, v_{s+1}) (\nu \cdot (v_{s+1} - v_i))_- dv dv_{s+1}.$$

Remark that for initial data of the form  $(f_0^{\otimes s})$ , the Boltzmann hierarchy is of the form  $(f^{\otimes s}(t))$ , where  $f(t)$  is solution of :

$$(24) \quad f(t) = T_1^0(t) f_0 + \int_0^t T_1^0(\tau) C_{1,2}^0 f(\tau)^{\otimes 2} d\tau$$

which is precisely the Boltzmann equation in the integral form.

To properly define this operator, we have to find a nice functional space on which the  $(f_0^s)$  will be defined :

*Definition 4.1.* For  $\beta > 0$  and  $\mu$  two constants, we define the Banach space  $X_{\beta,\mu}$  such that  $(f^s) \in X_{\beta,\mu}$  if and only if for all  $s \in \mathbb{N}^*$ ,  $f^s$  is measurable, symmetric, compatible:

$$(25) \quad \forall s \in \mathbb{N}^*, \forall \mathbf{z}_s \in (\Lambda \times \mathbb{R}^3)^s, f^s(\mathbf{z}_s) = \int_{\Lambda \times \mathbb{R}^3} f^{s+1}(\mathbf{z}_s, z_{s+1}) dz_{s+1}$$

and

$$(26) \quad \|(f^s)_s\|_{\beta,\mu} := \sup_{s \in \mathbb{N}^*} \operatorname{esssup}_{\mathbf{z}_s \in (\Lambda \times \mathbb{R}^3)^s} |f^s(\mathbf{z}_s)| \exp\left(\mu s + \frac{\beta}{2} \|\mathbf{v}_s\|^2\right)$$

is finite, with  $\|\mathbf{v}_s\|^2 = \sum_{i=1}^s |v_i|^2$ .

We denote the closed subspace of continuous functions

$$\tilde{X}_{\beta,\mu} := X_{\beta,\mu} \cap \prod_{s \geq 1} \mathcal{C}((\Lambda \times \mathbb{R}^3)^s).$$

**Theorem 4.1.** *For  $\beta > 0$ ,  $\mu$ , there exist  $\beta' > 0$ ,  $\mu'$  and a time  $T$  such that the Boltzmann hierarchy*

$$\begin{cases} \tilde{X}_{\beta,\mu} \rightarrow C([0, T], X_{\beta',\mu'}) \\ (f_0^s)_s \mapsto (f^s(t))_s \end{cases}$$

*is continuous.*

The rest of this section is devoted to the proof of the theorem.

In a first time we will prove that  $Q_{s,s+r}^0 f_0^{s+r}$  is well defined if  $(f_0^s)$  is in  $\tilde{X}_{\beta,\mu}$ . Then the continuity estimates of [14] and [21] show that the sum is well defined and continuous.

**4.2. Well-posedness of the operators  $Q_{s,s+r}^0(t)$ .** The main difficulty is that the transport semi-group  $T_1^0(t)$  does not send continuous functions onto continuous functions: discontinuities will appear in the future of  $\{0\}_t \times \partial\Lambda \times \mathbb{R}^3$ . Thus we have to check that we can apply  $C_{s,s+1}^0$  on the function  $T_{s+1}^0(t) f_0^{s+1}$ .

We will restrict ourselves to the case where  $f_0^s = g_1 \otimes \cdots \otimes g_s$ . We define the *future set*  $\mathcal{F}(t) \subset \mathbb{R}^+ \times \Lambda \times \mathbb{R}^3$  as

$$(27) \quad \mathcal{F}(t) := \{(\tau, x, v) | x - (\tau - t)v \in \partial\Lambda, \tau \geq t\}$$

and  $\mathcal{F}(t) \cap \{t = 0\} := \{(x, v) \in \mathbb{R}^3, (0, x, v) \in \mathcal{F}(t)\}$

Then we have the following lemma, equivalent to a "weak" propagation of continuity:

**Lemma 4.2.** *Let  $g$  be a measurable function on  $\Lambda \times \mathbb{R}^3$ , continuous outside  $(\mathcal{F}(t_1) \cup \cdots \cup \mathcal{F}(t_r)) \cap \{t = 0\}$  for  $(t_1, \dots, t_r) \in (\mathbb{R}_+^*)^s$  and dominated by  $e^{-\beta|v|^2/2}$  for some  $\beta > 0$ . Then  $T_1^0 g : (t, x, v) \mapsto (T_1^0(t)g)(x, v)$  is continuous outside  $\mathcal{F}(0) \cup \mathcal{F}(t_1) \cup \cdots \cup \mathcal{F}(t_r)$  and dominated by  $e^{-\beta|v|^2/2}$ .*

*Proof.* We will show that the trace of  $T_1^0 g$  on  $\Sigma^+ := \{(x, v) \in \partial\Lambda \times \mathbb{R}^3, v \cdot n(x) > 0\}$  is continuous. In the following we will denote  $t_0 := 0$ .

We define the function  $g_b$  on  $\mathbb{R}^+ \times \Sigma^+$  by :

$$g_b(t, x, v) := \int_{\mathbb{S}^2} g(x - t|v|\omega, |v|\omega)(\omega \cdot n(x))_- c_3 d\omega$$

with  $g$  extended by 0 outside  $\Lambda$ . A particle reflected with velocity  $|v|\omega$  crosses the domain in time  $1/|\omega \cdot \vec{e}_1||v|$  and  $g$  is continuous outside  $(\mathcal{F}(0) \cup \mathcal{F}(t_1) \cup \cdots \cup \mathcal{F}(t_r)) \cap \{t = 0\}$ . Thus the function  $\omega \mapsto g(x - t|v|\omega, |v|\omega)$  is continuous outside the set of directions  $\omega$  such that the particle meets the boundary at time  $t_i$  for some  $i$ :

$$\bigcup_{i=0}^r \left\{ \omega \in \mathbb{S}^2, \frac{1}{|\omega \cdot \vec{e}_1||v|} = (t - t_i) \right\}$$

which is of zero measure in  $\mathbb{S}^2$ . Thus  $g_b$  is continuous on  $\mathbb{R}^+ \times \Sigma^+$ . Now we define the operator  $B : C(\mathbb{R}^+ \times \Sigma^+) \rightarrow C(\mathbb{R}^+ \times \Sigma^+)$  which associates to  $f$  its image by the free flow after one more reflection:

$$Bf(t, x, v) := \int_{\mathbb{S}^2} f\left(t - \frac{1}{|\omega \cdot \vec{e}_1||v|}, x - \frac{\omega}{|\omega \cdot \vec{e}_1|}, |v|\omega\right) (n(x) \cdot \omega)_- c_3 d\omega,$$

where  $f$  is extended by 0 for  $t < 0$ . Using the same argument than for  $g_b$ ,  $Bf$  is continuous.

Applying the formula 8 in the case of one particle, we obtain that the trace of  $T_1^0 g$  on  $\mathbb{R}^+ \times \Sigma^+$  is

$$(28) \quad g_t := \sum_{i=0}^{\infty} B^i g_b.$$

We have to show that this sum converges normally on every compact set. In fact the sum is locally finite:

**Lemma 4.3.** *For  $(x, v) \in \Lambda \times \mathbb{R}^3$ , the one particle process has at most  $t|v|$  reflections.*

*Proof.* To have  $N$  reflections, a particle has to travel at least a distance  $N$ , so  $N \leq t|v|$ .  $\square$

From this bound we deduce that the series 28 is finite on every compact set, and so  $g_t$  is continuous.  $T_1 g$  is solution of the problem:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0 \\ f|_{\{0\} \times \Lambda \times \mathbb{R}^3} = g \\ f|_{\mathbb{R}_+ \times \Sigma^+} = g_t \end{cases}$$

where we fix all the boundary condition. Because  $g$  is continuous outside  $(\mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)) \cap \{t = 0\}$  and  $g_t$  is continuous outside  $\mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)$ ,  $T_1^0 g$  is continuous outside  $\mathcal{F}(t_0) \cup \dots \cup \mathcal{F}(t_r)$ .

The bound follows from maximum principle.  $\square$

Let  $f_1, \dots, f_s$  be measurable functions, continuous outside  $\mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)$  and dominated by  $Ae^{-\beta|v|^2/2}$  for some  $A > 0$ . Then  $T_s^0(t)(f_1 \otimes \dots \otimes f_s)$  is equal to  $(T_1^0(t)f_1) \otimes \dots \otimes (T_1^0(t)f_s)$  and

$$C_{s-1,s}^0(f_1 \otimes \dots \otimes f_s) = \sum_{i=1}^{s-1} f_1 \otimes \dots \otimes C_{1,2}^0(f_i \otimes f_s) \otimes \dots \otimes f_{s-1}.$$

If we fix  $t$  and  $x$ ,  $f_i$  has a discontinuity for  $v \in \bigcup_{i=1}^r \{u \in \mathbb{R}^3, (t, x, u) \in \mathcal{F}(t_i)\}$  the set of velocities such that the particles meet the boundary at time  $t_i$ , which is a union of planes of dimension 2. Using the "Carleman's collision parametrization" (see Appendix A),

$$\begin{aligned} C_{1,2}^0 f_1 \otimes f_2(t, x, v) &:= -f_1(t, x, v) \int_{\mathbb{R}^2 \times \mathbb{S}^2} f_2(t, x, v_*) ((v - v_*) \cdot \sigma)_- dv_* d\sigma \\ &\quad + \int_{\mathbb{R}^3 \times \mathbb{S}^2} f_1(t, x, v') f_2(t, x, v'_*) ((v - v_*) \cdot \sigma)_- dv_* d\sigma \\ &:= -f_1(t, x, v) \int_{\mathbb{R}^2 \times \mathbb{S}^2} f_2(t, x, v_*) ((v - v_*) \cdot \sigma)_- dv_* d\sigma \\ &\quad + \int_{\{(v', v'_*) \in \mathbb{R}^6, (v' - v) \cdot (v'_* - v) = 0\}} f_1(t, x, v') f_2(t, x, v'_*) dv' dS(v'_*) \end{aligned}$$

where  $dS(v'_*)$  is the Lebesgue measure on the affine plane  $\{v'_* \in \mathbb{R}^3, (v' - v) \cdot (v'_* - v) = 0\}$ . In the first term of the sum, we integrate  $f_2$  on a space of dimension 3, so the set of discontinuity is of zero measure, so the first term is continuous outside  $\mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)$ . For the second term, because we integrate  $v'$  on the full space, for almost all  $v'$ ,  $f_1(t, x, \cdot)$  is continuous at  $v'$  and  $v'_*$  lives in a plane transverse to the set of discontinuities of  $f_2(t, x, \cdot)$ . Thus  $f_1(t, x, v') f_2(t, x, v'_*)$  is integrable and the second term is continuous.

In addition we have the following bound:

$$\begin{aligned}
 (29) \quad & \left| C_{1,2}^0 f_1 \otimes f_2(t, x, v) \right| \\
 & \leq A^2 \int \left( e^{-\frac{\beta(|v|^2 + |v_*|^2)}{2}} + e^{-\frac{\beta(|v'|^2 + |v'_*|^2)}{2}} \right) ((v - v_*) \cdot \sigma)_- d\sigma dv_* \\
 & \leq CA^2 \int e^{-\frac{\beta(|v|^2 + |v_*|^2)}{2}} (|v| + |v_*|) dv_* \\
 & \leq A^2 C \beta^{-3/2} (\beta^{-1/2} + |v|) e^{-\beta|v|^2/2}
 \end{aligned}$$

for some constant  $A$ . Then for all  $\beta' < \beta$ , we can bound  $C_{1,2}^0 f_1 \otimes f_2$  by  $\tilde{C} e^{-\beta'|v|^2/2}$ . Finally we always have integrability with respect to  $v$ .

To summarize, if  $f_1, \dots, f_{s+1}$  are measurable functions, continuous outside  $(\mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)) \cap \{t = 0\}$  and dominated by  $C e^{-\beta|v|^2/2}$ , then

$$C_{s,s+1}^0 T_{s+1}^0(t)(f_1 \otimes \dots \otimes f_{s+1}) = \sum_{i=1}^s f_1^i \otimes \dots \otimes f_s^i$$

where the  $(f_j^i)_{i,j}$  are continuous outside  $\mathcal{F}(0) \cup \mathcal{F}(t_1) \cup \dots \cup \mathcal{F}(t_r)$  and dominated by  $\tilde{C} e^{-\beta'|v|^2/2}$  for all  $\beta' < \beta$ .

Iterating the process

$$(t_{s+1}, \dots, t_{s+r}) \mapsto T_s^0(t - t_{s+1}) C_{s,s+1}^0 \dots C_{s+r-1,s+r}^0 T_{s+r}^0(t_{s+r}) f^{\otimes(s+r)}$$

is equal to  $\sum_i f_1^i \otimes \dots \otimes f_s^i$  where the  $(f_j^i)_{i,j}$  are continuous outside  $\mathcal{F}(0) \cup \mathcal{F}(t_{s+1}) \cup \dots \cup \mathcal{F}(t_{s+r})$ . Fixing  $(t, \mathbf{x}_s, \mathbf{v}_s)$ , outside a finite number of  $(t_{s+1}, \dots, t_{s+r})$ , the  $f_j^i$  are continuous near  $\{(t_m, x_n, v_n), 1 \leq m \leq r, 1 \leq n \leq s\}$ . By integrating we get that  $Q_{s,s+r}^0(t) f^{\otimes(s+r)}$  is well defined on  $(\Lambda \times \mathbb{R}^3)^s$  and continuous outside  $\mathcal{F}(0)$ .

**Remark 4.1.** Note that if  $f_0$  verifies the boundary condition

$$(30) \quad \forall (x, v) \in \Sigma^+, f_0(x, v) = \int_{\mathbb{S}^2} f_0(x, \omega|v|) c_3(\Gamma(x)\omega \cdot \bar{\mathbf{e}}_1)_- d\omega,$$

$T_1^0 f_0$  is continuous, and then  $Q_{1,r}^0 f_0$  is always continuous. Finally we get:

**Proposition 4.4.** Let  $f_0$  be a continuous function on  $\Lambda \times \mathbb{R}^3$ , bounded by a Gaussian distribution and satisfying the condition 30. Then the solution  $f(t)$  of the Boltzmann equation with isotropic reflection is continuous.

### 4.3. Continuity estimates.

**Proposition 4.5.** There is a constant  $C$  independent of  $\beta, \mu$  and  $r \in \mathbb{N}$  such that

$$(31) \quad \left\| Q_{s,s+r}^0(t) f_0^{s+r} \right\|_{3\beta/4, \mu-1} \leq (C\beta^{-2} e^{-\mu t})^r \|f_{s+r}^0\|_{\beta, \mu},$$

$$(32) \quad \left\| Q_{s,s+r}^\epsilon(t) f_{0,N}^{(s+r)} \right\|_{3\beta/4, \mu-1} \leq (C\beta^{-2} e^{-\mu t})^r \|f_{0,N}^{(s)}\|_{\beta, \mu}.$$

*Proof.* We suppose  $f_0^{s+1}$  dominated by  $\exp(-\beta\|\mathbf{v}_{s+1}\|^2/2)$ . Because  $\exp(-\beta\|\mathbf{v}_s\|^2/2)$  is preserved by  $T_s^0(t)$  for all  $\beta > 0$ , using the bound 29,

$$\begin{aligned}
& |C_{s,s+1}T_{s+1}(t)f_0^{s+1}(\mathbf{z}_{s+1})| \\
& \leq C\beta^{-3/2} \left( s\beta^{-1/2} + \sum_{i=1}^s |v_i| \right) e^{-\frac{\beta}{2}\|\mathbf{v}_s\|^2} \\
& \leq C\beta^{-3/2} \left( s\beta^{-1/2} + \left( \sum_{i=1}^s |v_i| \right) e^{-(\beta-\beta')\|\mathbf{v}_s\|^2} \right) e^{-\frac{\beta'}{2}\|\mathbf{v}_s\|^2} \\
& \leq C\beta^{-3/2} \left( s\beta^{-1/2} + \frac{s^{1/2}}{(\beta-\beta')^{1/2}} ((\beta-\beta')\|\mathbf{v}_s\|^2)^{1/2} e^{-\frac{\beta-\beta'}{2}\|\mathbf{v}_s\|^2} \right) e^{-\frac{\beta'}{2}\|\mathbf{v}_s\|^2} \\
& \leq C\beta^{3/2} \left( s\beta^{-1/2} + \frac{s^{1/2}}{(\beta-\beta')^{1/2}} \right) e^{-\frac{\beta'}{2}\|\mathbf{v}_s\|^2}
\end{aligned}$$

using the Cauchy-Schwartz inequality between lines 3 and 4. Then iterating this bound and integrating on  $t > t_1 > \dots > t_r > 0$ , we get that for  $\|(f_0^s)\|_{\beta,\mu} \leq 1$

$$\begin{aligned}
& |Q_{s,s+r}^0(t)f_0^{s+r}(\mathbf{z}_s)| \\
& \leq \frac{t^r}{r!} \prod_{i=0}^{r-1} C\beta_{i+1}^{-3/2} \left( (s+i)\beta_{i+1}^{-1/2} + (s+i)^{1/2}(\beta_{i+1}-\beta_i)^{-1/2} \right) e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - \mu(r+s)}
\end{aligned}$$

for  $\beta = \beta_r > \dots > \beta_0 = 3\beta/4$ . For  $\beta_{i+1} - \beta_i = \beta/4r$ , we get :

$$\begin{aligned}
|Q_{s,s+r}^0(t)f_0^{s+r}(\mathbf{z}_s)| & \leq \frac{t^r(s+r)^r}{r!} (C\beta^{-2})^r e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - \mu(r+s)} \\
& \leq t^r e^{r+s} (C\beta^{-2})^r e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - s\mu} e^{-\mu r}
\end{aligned}$$

using the Stirling's formula. This is the expected estimate.

The same method works for  $Q_{s,s+r}^e(t)$ . □

We can sum these bounds and get

$$\|f^s(t)\|_{3\beta/4,\mu-1} \leq \frac{1}{1 - C\beta^{-2}e^{-\mu t}} \|(f_0^s)\|_{\beta,\mu}$$

which concludes the proof of Theorem 4.1. □

## 5. MAIN THEOREM

The main theorem of the paper is a weak convergence result. Indeed we will only look at the convergence of *observables* of the system, *i.e.* averaging with respect to the momentum variable. In addition, the marginals of the hard sphere system are only well defined in  $\mathcal{D}_\epsilon^s$ , so convergence will occur only away from the diagonal of the physical space. To summarize all these conditions, we define the following notion of convergence:

*Definition 5.1.* A sequence  $(f_N^s)_{1 \leq s \leq N}$  converges to a sequence  $(f^s)_s$  in average and locally uniformly off the diagonal if for all  $\varphi_s : \mathbb{R}^3 \rightarrow \mathbb{R}$ , continuous with polynomial growth at infinity, and for all compact sets  $K \subset \Lambda^s$ , away from the diagonal set  $\mathfrak{D}^s := \{\mathbf{x}_s \in \Lambda^s | \exists i, j, x_i = x_j\}$

$$(33) \quad I_{\varphi_s}(f_N^s - f^s)(\mathbf{x}_s) := \int_{(\mathbb{R}^3)^s} (f_N^s - f^s)(\mathbf{z}_s) \varphi_s(\mathbf{v}_s) d\mathbf{v}_s \xrightarrow{N \rightarrow \infty} 0 \text{ in } L^\infty(K).$$



**Theorem 5.1.** *Let  $\beta > 0$  and  $\mu$  be two constants. Then there exists a time  $T$  such that the following holds. Let  $(f_0^s)_s$  be an element of  $\tilde{X}_{\beta,\mu}$  and  $(W_{0,N})_N$  a sequence of symmetric functions in  $L^1 \cap L^\infty(\mathcal{D}_{1/\sqrt{N}}^N)$  with  $(f_{0,N}^{(s)})_{1 \leq s \leq N}$  their marginals. Then if  $(f_{0,N}^{(s)})$  converge to  $(f_0^s)$  as  $N \rightarrow \infty$  in norm  $\|\cdot\|_{\beta,\mu}$ , the sequence  $(f_N^{(s)}(t))_s$  solution to the BBGKY hierarchy converges in average locally uniformly to  $(f^s(t))_s$  the solution of the Boltzmann hierarchy with initial data  $(f_0^s)$ ,  $\forall t < T$ .*

**Remark 5.1.** *Note that the convergence of the observables is uniform for  $s = 1$ .*

*Proof.* As for the interacting case, we rewrite  $Q_{s,s+r}^0(t)$  with a characteristic formula. As in the case of interacting particles we construct for  $s, r \in \mathbb{N}$  the pseudotrajectory  $\zeta_{s,s+r}^0(\tau, t, \mathbf{z}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{r+s}, a, \boldsymbol{\sigma})$  where :

- $(a, \boldsymbol{\sigma}) \in \mathfrak{A}_s^{s+r} \times \{\pm 1\}^r$ ,
- $\tau \in [0, t]$ ,
- $(\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r} := (t_i, \nu_i, \bar{v}_i)_{s < i \leq s+r} \in (\mathbb{R} \times \mathbb{S}^2 \times \mathbb{R}^3)^r$  with  $t > t_{s+1} \cdots > t_{s+r} > 0$ ,
- $\bar{\boldsymbol{\omega}}_{r+s} = (\bar{\omega}_1, \dots, \bar{\omega}_{r+s}) \in \Omega^{r+s}$ .

The number of particles of  $\zeta_{s,s+r}^0(\tau)$  is  $s+k$  for  $\tau$  between  $t_k$  and  $t_{k+1}$  and  $\bar{\boldsymbol{\omega}}_{r+s}^0(\tau)$  the evolution of the reflection parameters. We define  $(\zeta_{s,s+r}^0(\tau), \bar{\boldsymbol{\omega}}_{r+s}^0(\tau))$  by

- $(\zeta_{s,s+r}^0, \bar{\boldsymbol{\omega}}_{r+s}^0)(t) := (\mathbf{z}_s, \bar{\boldsymbol{\omega}}_{s+r})$ ,
- for  $\tau \in (t_{k+1}, t_k)$ ,  $\bar{\boldsymbol{\omega}}_{r+s}^0(\tau)$  is constant and  $(\zeta_{s,s+r}^0(\tau), \bar{\boldsymbol{\omega}}_{r+s}^0(\tau))$  follow the hard sphere backward flow described at section 2.1,
- at time  $t_k^+$ , a particle is added at position  $x_k^0(t_k) = x_{a(k)}^0(t_k)$  and velocity  $\bar{v}_k$ . If  $\sigma = 1$ , we have  $(\bar{v}_k - v_{a(k)}(t_k^+)) \cdot \nu_k > 0$  and we apply the scattering of velocities. Else if  $\sigma_k = -1$ , we do not apply scattering.

We denote  $\mathcal{G}^0(\mathbf{z}_s, t, a, \boldsymbol{\sigma})$  the set of coordinates, *i.e.* the  $((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{r+s})$  such that the pseudo-trajectory follows the collision parameters  $(a, \boldsymbol{\sigma})$ , and

$$\begin{aligned} & d\Lambda_{a,\boldsymbol{\sigma}}^0((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{r+s}) \\ &= \left( \prod_{k=s+1}^{s+r} \left[ \sigma_k \nu_k \cdot (\bar{v}_k - v_{a(k)}^0(t_k^+)) \right]_+ dt_k d\bar{v}_k d\nu_k \right) d\mathbb{P}^{s+r}(\bar{\boldsymbol{\omega}}_{r+s}). \end{aligned}$$

Then we can rewrite  $Q_{s,s+r}^0(t) f^{s+r}(\mathbf{z}_s)$  as

$$(34) \quad \begin{aligned} & Q_{s,s+r}^0(t) f^{s+r}(\mathbf{z}_s) \\ &:= \sum_{a,\boldsymbol{\sigma}} \sigma_{s+1} \cdots \sigma_{s+r} \int_{\mathcal{G}^0(\mathbf{z}_s, t, a, \boldsymbol{\sigma}_r)} d\Lambda_{a,\boldsymbol{\sigma}_r}^0((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{r+s}) f_0^{s+r}(\zeta^0(0)). \end{aligned}$$

We shall prove the one-to-one convergence

$$I_{\varphi_s} \left( \int_{\mathcal{G}^\epsilon(\mathbf{z}_s, t, a, \boldsymbol{\sigma})} d\Lambda_{a,\boldsymbol{\sigma}}^\epsilon f_{0,N}^{(s+r)}(\zeta^\epsilon(0)) \right) \xrightarrow{N \rightarrow \infty} I_{\varphi_s} \left( \int_{\mathcal{G}^0(\mathbf{z}_s, t, a, \boldsymbol{\sigma})} d\Lambda_{a,\boldsymbol{\sigma}}^0 f_0^{s+r}(\zeta^0(0)) \right)$$

at all points  $(t, \mathbf{x}_s)$  and all collision parameters  $(a, \boldsymbol{\sigma}_r)$ , using that the pseudotrajectories  $\zeta^\epsilon$  and  $\zeta^0$  are in average not so far. There are two reasons producing a big error between the two trajectories. The first one is the *recollisions* (collision of two particles that have already been created) that can occur for the interacting process but not for the punctual particles. The second comes from the difference between  $\bar{\boldsymbol{\omega}}^\epsilon(0)$  and  $\bar{\boldsymbol{\omega}}^0(0)$ : because there is a shift between  $\zeta^0(\tau)$  and  $\zeta^\epsilon(\tau)$ , the times of

reflection are not exactly the same. Thus there exists a set of times during which  $\omega^\epsilon(\tau)$  is different from  $\omega^0(\tau)$ . If 0 is in this set the final velocity will be very different.

To make the convergence term by term work, we have to perform some truncation. First we note that the continuity bound gives

$$(35) \quad \mathcal{R}_1 := \left| \sum_{r=0}^{N-s} Q_{s,s+r}^\epsilon (f_{0,N}^{(s+r)} - f_0^{s+r}) \right| \leq C_{\beta,\mu} e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - (\mu-1)s} \left\| (f_{0,N}^{(s+r)} - f_0^{s+r})_{1 \leq s \leq N} \right\|_{\beta,\mu}$$

and we have to look only at the semigroups applied to  $(f_0^s)_s$ .

Then we have to bound the number of creations of particles by an integer  $R$  that depends on  $N$ . The rest will be bounded by :

$$(36) \quad \mathcal{R}_2 := \left| \sum_{r>R} Q_{s,s+r}^0(t) f_0^{s+r} \right| \leq \sum_{r>R} (C_{\beta,\mu} t)^r e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - (\mu-1)s} \|(f_0^s)_s\|_{\beta,\mu} \leq C 2^{-R} e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - (\mu-1)s} \|(f_0^s)_s\|_{\beta,\mu}$$

for  $t \leq 1/(2C_{\beta,\mu}) =: T$ . The same estimates hold for the BBGKY hierarchy.

Next we have to cut the high energy. Because  $(f_0^s)_s$  is bounded in  $X_{\beta,\mu}$ ,  $(f_0^s \mathbb{1}_{\|\mathbf{v}_i\|^2 > E})_s$  is bounded in  $X_{7\beta/8,\mu}$  by  $e^{-7\beta E/16} \|(f_0^s)_s\|_{\beta,\mu}$ . Thus we have the bound (the same bound holds for BBGKY)

$$(37) \quad \mathcal{R}_3 := \left| \sum_{r \geq 0} Q_{s,s+r}^0(t) (f_0^{s+r} \mathbb{1}_{\|\mathbf{v}_{s+r}\|^2 > E}) \right| \leq C e^{-\beta E/16} e^{-\frac{3\beta}{8}\|\mathbf{v}_s\|^2 - (\mu-1)s} \|(f_0^s)_s\|_{\beta,\mu}.$$

Thus for any  $\eta > 0$  we can fix  $R, E$  and  $N_0$  such that for  $N > N_0$ ,

$$\|\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3\|_{3\beta/4,\mu-1} \leq \eta.$$

From this  $\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$  converges to zero in the sense of Definition 5.1 when  $R, E$  and  $N_0$  tend to infinity, uniformly in  $\epsilon$ .

Therefore the proof is reduced to the proof of the following proposition:

**Proposition 5.2.** *For all compact sets  $K \subset \Lambda^s \setminus \mathcal{D}_s$ , all collision parameters  $(a, \sigma)$  and all test functions  $\varphi_s$ ,*

$$(38) \quad \int_{\mathbb{G}^\epsilon(t, \mathbf{x}_s, a, \sigma, E)} d\mathbf{v}_s d\Lambda_{a,\sigma}^\epsilon f_0^{s+r}(\zeta^\epsilon(0)) \varphi_s(\mathbf{v}_s) \xrightarrow{N \rightarrow \infty} \int_{\mathbb{G}^0(t, \mathbf{x}_s, a, \sigma, E)} d\mathbf{v}_s d\Lambda_{a,\sigma}^0 f_0^{s+r}(\zeta^0(0)) \varphi_s(\mathbf{v}_s)$$

uniformly on  $[0, T] \times K$ , where  $\mathbb{G}^\epsilon(t, \mathbf{x}_s, a, \sigma, E)$  is the set of parameters

$$\left\{ (\mathbf{v}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{s+r}), \mathbf{v}_s \in \mathbb{R}^{3s}, \right. \\ \left. ((\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1,s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \in \mathcal{G}^\epsilon(t, \mathbf{x}_s, \mathbf{v}_s, a, \sigma), \|\mathbf{v}_s\|^2 + \|\bar{\mathbf{v}}_{s+1,s+r}\|^2 \leq E \right\},$$

and same definition for  $\mathbb{G}^0(t, \mathbf{x}_s, a, \sigma, E)$ .

□

## 6. PROOF OF PROPOSITION 5.2

We fix collision parameters  $(a, \sigma)$  and a compact set  $K \in \Lambda^s \setminus \mathfrak{D}^s$ . In order to prove 38, we compare the two pseudotrajectories. Fixed an initial position  $\mathbf{x}_s$  at time  $t$ , we need to know the first difference between hard spheres and punctual process (going backward from time  $t$  to 0). We can construct four bad sets  $\mathcal{P}_1(\epsilon)$ ,  $\mathcal{P}_2(\epsilon)$ ,  $\mathcal{P}'_2(\epsilon)$  and  $\mathcal{P}_3(\epsilon)$  such that

- for  $(\mathbf{v}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \in \mathcal{P}_1(\epsilon)$ , there is first a *shift*, *i.e.* there is a  $i \in \{s+1, \dots, s+r\}$  such that  $\bar{\boldsymbol{\omega}}_{s+r}^0(t_i) \neq \bar{\boldsymbol{\omega}}_{s+r}^\epsilon(t_i)$  and there is no recollision during  $(t_i, t)$  (recall that  $t_{s+r} = 0$ ),
- for  $(\mathbf{v}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \in \mathcal{P}_2(\epsilon)$ , the hard sphere process has first a recollision, *i.e.* there is a first recollision at time  $\tau$  and no shift during  $(\tau, t)$ ,
- for  $(\mathbf{v}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \in \mathcal{P}'_2(\epsilon)$ , the punctual process has first an *overlap*, *i.e.* two particles reach a distance less than  $\epsilon$  at some time  $\tau$  (with an exception for particles  $i$  and  $a(i)$  just after  $t_i$ ),
- for  $(\mathbf{v}_s, (\mathbf{t}, \boldsymbol{\nu}, \bar{\mathbf{v}})_{s+1, s+r}, \bar{\boldsymbol{\omega}}_{s+r}) \in \mathcal{P}_3(\epsilon)$ , there is some  $i \in \{s+1, \dots, s+r\}$  such that  $x_{a(i)}^\epsilon(t_i)$  is at distance less than  $2\epsilon$  from the boundary or from any other particle, and  $\mathcal{P}_3(\epsilon) \cap (\mathcal{P}_1(\epsilon) \cup \mathcal{P}_2(\epsilon) \cup \mathcal{P}'_2(\epsilon)) = \emptyset$ .

Note that  $\mathbb{G}^\epsilon \setminus (\mathcal{P}_1(\epsilon) \cup \mathcal{P}_2(\epsilon) \cup \mathcal{P}'_2(\epsilon) \cup \mathcal{P}_3(\epsilon)) = \mathbb{G}^0 \setminus (\mathcal{P}_1(\epsilon) \cup \mathcal{P}_2(\epsilon) \cup \mathcal{P}'_2(\epsilon) \cup \mathcal{P}_3(\epsilon))$ . Indeed outside  $\mathcal{P}_3$ , all relative positions  $\nu_i$  are allowed at creation  $i$ , and outside  $\mathcal{P}_1(\epsilon) \cup \mathcal{P}_2(\epsilon) \cup \mathcal{P}'_2(\epsilon)$  the velocities of the incoming particles at the reflections are the same in hard sphere and punctual backward pseudotrajectories.

In the following we will need to integrate on the collision parameters, we then introduce the following useful terminology. The particle  $a(i)$  is *deviated* by the creation  $i \in \{s+1, \dots, s+r\}$  if  $\sigma_i$  is positive. In that case there is a scattering of the velocity. The particle  $i$  is by definition deviated by the creation  $i$ . We recall that we look at backward trajectories and we follow time in the inverse sense.

**6.1. Continuity estimates of the two pseudotrajectories.** We begin by a continuity estimate for the process involving one particle.

**Lemma 6.1.** *Let  $(x^0, \tilde{x}^0) \in \Lambda^2$ ,  $v \in B(E^{1/2})$  (the ball in  $\mathbb{R}^3$  of diameter  $E^{1/2}$ ) and  $\bar{\boldsymbol{\omega}}_2^0 \in \Omega^2$ . For any collision parameters  $(v_*, \nu) \in \mathbb{S}^2 \times B(E^{1/2})$ , we define two 2-punctual particles backward processes  $(\mathbf{z}_2(\tau), \bar{\boldsymbol{\omega}}_2(\tau))$  and  $(\tilde{\mathbf{z}}_2(\tau), \tilde{\bar{\boldsymbol{\omega}}}_2(\tau))$  for  $\tau \in (0, t)$ , with initial data at  $\tau = t$*

- $(x^0, v, x^0, v_*, \bar{\boldsymbol{\omega}}_2)$  (respectively  $(x^0, v, x^0 + \epsilon\nu, v_*, \bar{\boldsymbol{\omega}}_2^0)$ ) if  $(v - v_*) \cdot \nu < 0$ ,
- $(x^0, v', x^0, v'_*, \bar{\boldsymbol{\omega}}_2^0)$  (respectively  $(x^0, v', x^0 + \epsilon\nu, v'_*, \bar{\boldsymbol{\omega}}_2^0)$ ) if  $(v - v_*) \cdot \nu > 0$ , with  $(v', v'_*)$  given by the scattering rule 4.

*Then if the particle  $i \in \{1, 2\}$  is deviated by the creation, there exists a constant  $C > 0$  such that for  $\tau \in (0, t)$*

$$(39) \quad \int_{B(E^{1/2}) \times \mathbb{S}^2} |x_i(\tau) - \tilde{x}_i(\tau)| \mathbb{1}_{\bar{\boldsymbol{\omega}}_i(\tau) = \tilde{\bar{\boldsymbol{\omega}}}_i(\tau)} |\nu \cdot (v - v_*)| dv_* d\nu \\ \leq CE^{5/2} (\epsilon + |x_0 - \tilde{x}_0|) |\log(\epsilon + |x_0 - \tilde{x}_0|)|.$$

*Proof.* Fix  $(v_*, \nu)$  and denote  $\tau_k$  (respectively  $\tau_k + \delta\tau_k$ ) the time of the  $k$ -th reflection of the particle  $z_i$  (respectively the particle  $\tilde{z}_i$ ). For  $k \geq 1$ , because between their  $k$ -th and  $(k+1)$ -th reflections the two particles have the same velocity and the two sides of  $\partial\Lambda$  are parallel,

$$(x_i(\tau_k), x_i(\tau_{k+1}), \tilde{x}_i(\tau_{k+1} + \delta\tau_{k+1}), \tilde{x}_i(\tau_k + \delta\tau_k))$$

forms a parallelogram. Hence  $\delta\tau_k$  and  $(\tilde{x}_i(\tau_k + \delta\tau_k) - x_i(\tau_k))$  do not depend on  $k$  (we denote them  $\delta\tau$  and  $\delta x$ ).

Because we suppose that there is no shift at time  $\tau$ ,  $z_i$  and  $\tilde{z}_i$  have the same number of reflections. If there is no reflection,  $|x_i(\tau) - \tilde{x}_i(\tau)|$  is constant. If there are  $k \geq 1$  reflections, we can suppose that  $\delta\tau > 0$  and

$$\begin{aligned} |x_i(\tau) - \tilde{x}_i(\tau)| &= |x_i(\tau_k) - \tilde{x}_i(\tau_k)| \\ &\leq |x_i(\tau_k) - \tilde{x}_i(\tau_k + \delta\tau)| + |\tilde{x}_i(\tau_k + \delta\tau) - \tilde{x}_i(\tau_k)| \\ &\leq |\delta x| + E^{1/2}|\delta\tau|. \end{aligned}$$

We can treat similarly the case  $\delta\tau < 0$ .

Because the two trajectories are parallel before the first reflection,

$$\delta\tau = ([x_i(0) - \tilde{x}_i(0)] \cdot \mathbf{e}_1) / (v_i \cdot \mathbf{e}_1) \text{ and } |\delta x| \leq \frac{|v_i|}{|v_i \cdot \mathbf{e}_1|} |x_i(0) - \tilde{x}_i(0)|.$$

Thus if  $\bar{\omega}_i(\tau) = \tilde{\omega}_i(\tau)$ ,

$$|x_i(\tau) - \tilde{x}_i(\tau)| \leq |x_i(0) - \tilde{x}_i(0)| \left( 1 + \frac{2E^{1/2}}{|v_i \cdot \mathbf{e}_1|} \right) \leq (|x_0 - \tilde{x}_0| + \epsilon) \frac{3E^{1/2}}{|v_i \cdot \mathbf{e}_1|},$$

and it is also bounded by some constant  $C_\Lambda$  because  $\Lambda$  is compact. Because the particle  $i$  is deviated, we can integrate on the scattering: using estimation 47 (Appendix A)

$$\begin{aligned} &\int_{B(E^{1/2}) \times \mathbb{S}^2} |x_i(\tau) - \tilde{x}_i(\tau)| \mathbb{1}_{\bar{\omega}_i(\tau) = \tilde{\omega}_i(\tau)} |\nu \cdot (v - v_*)| dv_* dv \\ &\leq \int_{B(E^{1/2}) \times \mathbb{S}^2} \left( \frac{3E^{1/2}(|x_0 - \tilde{x}_0| + \epsilon)}{|v_i \cdot \mathbf{e}_1|} \wedge C_\Lambda \right) |\nu \cdot (v - v_*)| dv_* dv \\ &\leq CE^{5/2}(|x_0 - \tilde{x}_0| + \epsilon) \log(|x_0 - \tilde{x}_0| + \epsilon). \end{aligned}$$

□

Fix  $(t_{s+1}, \dots)$ . Let  $\tau \in [0, t] \setminus \{t_i, s < i < s+r\}$ . If there is no recollision nor overlap and if  $\bar{\omega}^\epsilon(\hat{t}) = \bar{\omega}^0(\hat{t})$  for  $\hat{t} \in \mathfrak{T}(\tau) := \{\tau\} \cup \{t_i, t_i > \tau\}$ , then by an iteration argument  $v_k^0(\tau) = v_k^\epsilon(\tau)$  (in the following denoted  $v_k(\tau)$ ). Fixing  $\mathbf{x}_s, \mathbf{v}_s, \mathbf{t}_{s+1, s+r}$  and  $\bar{\omega}_{s+r}$ , we can then iterate the previous estimates and obtain

**Proposition 6.2.** *For  $k \in \{1, \dots, r\}$ ,  $\tau \in [t_{k+1}, t_k)$  and  $i \in \{1, \dots, s+k\}$  fixed, we have that:*

$$(40) \quad \begin{aligned} &\int_{B_k(E^{1/2}) \times (\mathbb{S}^2)^k} \mathbb{1}_{\text{no recollision}} \mathbb{1}_{\bar{\omega}^\epsilon(\hat{t}) = \bar{\omega}^0(\hat{t}), \hat{t} \in \mathfrak{T}(\tau)} |x_i^0(\tau) - x_i^\epsilon(\tau)| \\ &\quad \times \prod_{l=s+1}^{s+k} |\nu_l \cdot (\bar{v}_l - v_{a(l)}(t_l^+))| d\nu_l d\bar{v}_l \leq \epsilon k \left( CE^{5/2} |\log \epsilon| \right)^k \end{aligned}$$

for  $B_k$  the ball on  $\mathbb{R}^{3k}$ .

*Proof.* Note that  $B_k(E^{1/2}) \subset (B_1(E^{1/2}))^k$ , thus we will replace the first set by the second one.

In the following  $\tilde{i}$  will indicate the following pseudoparticle:  $\tilde{i}$  is equal to  $i$  from time  $\tau$  to time  $t_i$ . Then  $\tilde{i}$  becomes  $a(i)$  until time  $t_{a(i)}$ , etc until  $t$ . We can suppose that every creation deviates  $\tilde{i}$ . Else it does not influence  $|x_i^0(\tau) - x_i^\epsilon(\tau)|$  and we can count it as a factor  $CE^2$ .

We want to prove recursively for  $m \in \{1, \dots, k\}$

$$\begin{aligned}
 (41) \quad & \int_{(B(E^{1/2}) \times \mathbb{S}^2)^m} \mathbb{1}_{\text{no recollision}} \mathbb{1}_{\omega^\epsilon(\dot{t}) = \omega^0(\dot{t})} |x_i^0(\tau) - x_i^\epsilon(\tau)| \\
 & \times \prod_{l=s+k+1-m}^{s+k} |\nu_l \cdot (\bar{v}_l - v_{a(l)}(t_l^+))| d\nu_l d\bar{v}_l \\
 & \leq (CE^{5/2})^m |\log \epsilon|^{m-1} [(m-1)\epsilon + \eta (|x_i^0(t_{s+k+1-m}^+) - x_i^\epsilon(t_{s+k+1-m}^+)| + \epsilon)]
 \end{aligned}$$

for  $\eta(x) = x|\log(x)|$ . The initialization is provided by the previous lemma. To prove the induction, as in the previous lemma

$$\begin{aligned}
 & |x_i^0(t_{s+k+1-m}^+) - x_i^\epsilon(t_{s+k+1-m}^+)| \\
 & \leq \left( |x_i^0(t_{s+k-m}^-) - x_i^\epsilon(t_{s+k-m}^-)| \frac{3E^{1/2}}{|v_i(t_{s+k-m}^-) \cdot \bar{\mathbf{e}}_1|} \right) \wedge C_\Lambda.
 \end{aligned}$$

Thus  $(|x_i^0(t_{s+k+1-m}^+) - x_i^\epsilon(t_{s+k+1-m}^+)| + \epsilon)$  stays between  $\epsilon$  and  $C_\Lambda + \epsilon$  and its log is smaller than  $\sup(|\log \epsilon|, |\log C_\Lambda + \epsilon|) = |\log \epsilon|$ . Hence the left hand side of 41 is bounded by

$$(CE^{5/2})^m |\log \epsilon|^m \left[ m\epsilon + \left( |x_i^0(t_{s+k-m}^-) - x_i^\epsilon(t_{s+k-m}^-)| \frac{3E^{1/2}}{|v_i(t_{s+k-m}^-) \cdot \bar{\mathbf{e}}_1|} \right) \wedge C_\Lambda \right].$$

With the help of 47, we can integrate this estimates on  $B(E^{1/2}) \times \mathbb{S}^2$  with respect to the measure

$$|\nu_{s+k-m} \cdot (\bar{v}_{s+k-m} - v_{a(s+k-m)}(t_{s+k-m}^+))| d\nu_{s+k-m} d\bar{v}_{s+k-m}$$

and obtain the expected estimates.

Before the first creation punctual and hard sphere backward processes coincide, hence  $|x_i^0(t_{s+1}^+) - x_i^\epsilon(t_{s+1}^+)|$  vanishes. Therefore  $m = k$  gives the expected result.  $\square$

Integrating 40 on  $(\mathbf{v}_s, \mathbf{t}_{s+1, s+r}, \bar{\omega}_{s+r})$ ,

$$(42) \quad \int_{\mathbb{G}^0 \setminus (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}'_2 \cup \mathcal{P}_3)(\epsilon)} |\zeta^\epsilon(0) - \zeta^0(0)| d\mathbf{v}_s d\Lambda^0 \leq \epsilon |\log \epsilon|^r \frac{(CE^{5/2})^{s+r} T^r}{r!}.$$

Note that if the domain is the torus, the previous proof provides the bound  $\epsilon (CE^{5/2})^{s+r} T^r / r!$ . The factor  $|\log \epsilon|^r$  originates from the grazing reflections, and the author does not believe that this continuity estimate can be improved considerably.

Because  $f_0^{s+r}$  is continuous and  $(\Lambda \times B(E^{1/2}))^{s+r}$  is compact,  $f_0^{s+r}$  is uniformly continuous and the error goes to zero. Moreover  $\varphi_s(\mathbf{v}_s)$  is bounded on  $\mathbb{G}^0$ . Finally we get:

**Proposition 6.3.**

$$(43) \quad \int_{\mathbb{G}^0 \setminus (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}'_2 \cup \mathcal{P}_3)(\epsilon)} [f_0^{s+r}(\zeta^\epsilon(0)) - f_0^{s+r}(\zeta^0(0))] \varphi_s(\mathbf{v}_s) d\mathbf{v}_s d\Lambda^0 \rightarrow 0$$

uniformly in  $[0, T] \times K$ .



Using that the surface of  $\{\omega \in \mathbb{S}^2, \omega \cdot \bar{\mathbf{e}}_1 \in [a, b]\}$  is smaller than  $2\pi((b-a) \wedge 2)$ ,

$$\begin{aligned} & \int_{[0, T] \times \mathbb{S}^2} \left( \mathbb{1}_{t_{i'} \in [\tau_k - |\delta\tau|, \tau_k + |\delta\tau|]} + \mathbb{1}_{|\omega_j^k \cdot \bar{\mathbf{e}}_1| \in \left[ \frac{1}{|v|((\tau_k - t_{i'}) + |\delta\tau|)}, \frac{1}{|v|((\tau_k - t_{i'}) - |\delta\tau|)} \right]} \right) (\omega_j^k \cdot \bar{\mathbf{e}}_1)_+ \\ & \quad \times dt_{i'} d\omega_j^k \\ & \leq C|\delta\tau| + C \int_0^T \left( \mathbb{1}_{t_{i'} \notin [\tau_k - |\delta\tau|, \tau_k + |\delta\tau|]} \frac{|\delta\tau|}{|v|(t_{i'} - \tau_k - |\delta\tau|)(t_{i'} - \tau_k + |\delta\tau|)} \wedge 1 \right) dt_{i'} \\ & \leq C|\delta\tau| + 2C|\delta\tau| \int_1^{T/|\delta\tau|} \frac{1}{|v|(s^2 - 1)} \wedge 1 ds \\ & \leq C|\delta\tau| + \frac{2C|\delta\tau|}{|v|} \leq C|\delta\tau| \left( 1 + \frac{2}{|v \cdot \bar{\mathbf{e}}_1|} \right). \end{aligned}$$

making the change of variable  $s = \pm \frac{t_{i'} - \tau_k}{|\delta\tau|}$ .

**In the case where only one particle has a reflection in  $(t_{i'}, t_i)$ ,**  $t_{i'}$  has to stay in  $[\tau_1 - |\delta\tau|, \tau_1 + |\delta\tau|]$ , and thus in a set of size  $2|\delta\tau|$ . We sum on all possible "last reflections". There are at most  $E^{1/2}T$  reflections and as in the previous section  $|\delta\tau| \leq (\epsilon + |x_{a(i)}^0(t_i) - x_{a(i)}^\epsilon(t_i)|) / (|v \cdot \bar{\mathbf{e}}_1|)$ . Finally the set of parameters  $(t_{i'}, (\omega_j^k)_{k < k_j(t_i)})$  such that there is a shift is of size at most

$$CT \left( \left( \epsilon + |x_{a(i)}^0(t_i) - x_{a(i)}^\epsilon(t_i)| \right) \left( \frac{1}{|v \cdot \bar{\mathbf{e}}_1|} + \frac{1}{|v \cdot \bar{\mathbf{e}}_1|^2} \right) \wedge 1 \right).$$

Integrating over  $(\nu_i, \bar{v}_i)$ , and applying 47 and 48, the set of parameters  $((\nu_i, \bar{v}_i), t_{i'}, (\omega_j^k)_{k < k_j(t_i)})$  is of size at most

$$\left( \epsilon + |x_{a(i)}^0(t_i) - x_{a(i)}^\epsilon(t_i)| \right)^{1/2} C(E, T).$$

Combining this estimation, the estimation of  $|x_{a(i)}^0(t_i) - x_{a(i)}^\epsilon(t_i)|$  in 40 and the estimation of the remaining term,  $|\mathcal{P}_1^{i, j, i'}|$  converges uniformly to 0.  $\square$

**6.3. Estimation of  $\mathcal{P}_2 \cup \mathcal{P}_2' \cup \mathcal{P}_3$ .** In the following we look only at the hard sphere process and we drop the exponents  $\epsilon$ .

We focus on  $|\mathcal{P}_2|$ ; the size of  $\mathcal{P}_2'$  and  $\mathcal{P}_3$  can be estimated similarly.

We begin by cutting the grazing velocities: we consider the set of initial parameters  $\mathcal{P}_4$  such that

- for  $i \in \{1, \dots, s\}$ ,  $|v_i \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$
- for  $i \in \{s+1, \dots, s+r\}$ , if the particle  $j$  is deviated at time  $t_i$ ,  $|v_j(t_i^-) \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$ ,  $\forall k \in \{1, \dots, i\} \setminus \{j\}$ ,  $|(v_j(t_i^+) - v_k(t_i^+)) \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$ ,
- for any reflection of a particle  $i$  at time  $\tau$ , the reflected velocity of reflection  $v_i(\tau^-) := \omega_i^l |v_i(\tau^+)|$  has to verify  $|v_i(\tau^-) \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$ ,  $\forall k \neq i$ ,  $|v_j(\tau^-) - v_k(\tau^-)| > \epsilon^{1/4}$ .

Because velocities are bounded by  $E^{1/2}$ , there are at most  $2(s+r)E^{1/2}T$  reflections in the interval  $[0, t]$  and the last condition deals only with a finite number of reflection parameters in  $\bar{\omega}_{s+r}$ . Hence the size of the set  $\mathbb{G}^\epsilon \setminus \mathcal{P}_4$  goes to zero as  $\epsilon$  goes to 0. We restrict to  $\mathcal{P}_4$  from now on. We split  $\mathcal{P}_2 \cap \mathcal{P}_4$  into the partition  $(\mathcal{P}_{i,j})_{1 \leq i < j \leq s+r}$  where  $i$  and  $j$  are the two first particles that collide.

We denote  $k$  the last creation such that the particles  $i$  or  $j$  are deviated (it depends only on collision parameters  $(a, \sigma)$ ).

**We consider first the case where only one particle (say  $i$ ) is involved in the creation  $k$ .** To deal with the periodicity of  $\Lambda$ , we consider the covering  $\tilde{\Lambda} := [0, 1] \times \mathbb{R} \xrightarrow{p} [0, 1] \times \mathbb{T}^2 = \Lambda$ . Particles  $i$ ,  $j$  and  $k$  have infinitely many copies  $(i_{\vec{k}})$ ,  $(j_{\vec{k}})$ ,  $(k_{\vec{k}})$ , with coordinates  $(x_{i_{\vec{k}}}(t), v_{i_{\vec{k}}}(t)) := (x_i(t) + \vec{k}, v_i(t))$  for  $\vec{k} \in \{0\} \times \mathbb{Z}^2$ . If some  $i_{\vec{k}}$  has a recollision with a particle  $j_{\vec{k}'}$  in  $\tilde{\Lambda}$ , then the particles  $i$  and  $j$  have a recollision in  $\Lambda$ . Because  $v_i$  and  $v_j$  are bounded by  $E^{1/2}$ , the particle  $i_{\vec{0}}$  can interact only with the  $j_{\vec{k}}$  for  $|\vec{k}| \leq 2TE^{1/2}$ . In the following we denote  $x_i := x_{i_{\vec{0}}}$ ,  $x_k := x_{k_{\vec{0}}}$  and  $x_j := x_{j_{\vec{0}}}$ .

First we cut the reflection times  $t_k$  such that  $|x_i(t_k) - x_j(t_k)| \leq \epsilon^{1/3}$  and  $d(x_i(t_k), \partial\tilde{\Lambda}) < \epsilon^{1/3}$ . We observe that  $x_k$  and  $x_j$  are polygonal trajectories, with at most  $TE^{1/2}$  branches. Thus the two particles can approach each other at most  $T^2E$  times at distance less than  $\epsilon^{1/3} + \epsilon$ . Thanks to the condition on  $|v_i(t_k^-) \cdot \vec{e}_1|$  and  $|v_k(t_k^-) - v_j(t_k^-)|$ , for  $t_k$  outside a set of size  $CT^2E\epsilon^{1/3-1/4}$ , the particle  $k$  does not come close to the boundary nor to the particle  $j$ .

Then we consider the "virtual" particle  $\tilde{j}$  as a particle which moves along straight lines and coincides with  $j$  after its last reflection before the recollision, and we denote  $(x_{\tilde{j}}(t), v_{\tilde{j}}) := (x_j(\tau_j^-) + (t - \tau_j)v_j(\tau_j^-), v_j(\tau_j^-))$  its coordinates, for  $\tau_j$  the time of the last reflection of  $j$  before the recollision. If there is a recollision between  $i$  and  $j$ , there is a recollision between  $i$  and  $\tilde{j}$ . Because  $d(x_i(t_k), \partial\tilde{\Lambda})$  and  $|x_i(t_k) - x_j(t_k)|$  are greater than  $\epsilon^{1/3}$ ,  $|x_j(t_k) - x_i(t_k)| > \epsilon^{1/3}$ .

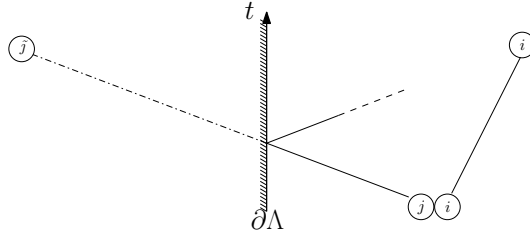


FIGURE 5. Virtual particle.

If  $i$  has no reflection until the recollision, then there exists a time  $\tau \in (0, t_k)$  and a direction  $\nu_{\text{rec}} \in \mathbb{S}^2$  such that :

$$(x_j(t_k) - x_i(t_k)) + (\tau - t_k)(v_j - v_i(t_k^-)) = \epsilon \nu_{\text{rec}}.$$

Thus  $(v_i(t_k^-) - v_j)$  is in a cone  $\mathfrak{C}[(x_j(t_k) - x_i(t_k)), \alpha]$  of axes  $(x_j(t_k) - x_i(t_k))$  and angle  $\alpha := 2 \arcsin(\epsilon/|x_j(t_k) - x_i(t_k)|)$ . Because  $|x_j(t_k) - x_i(t_k)| > \epsilon^{1/3}$  and  $|v_i(t_k^-)| < E^{1/2}$ ,  $v_i(t_k^-)$  has to be in a rectangle of size  $E^{1/2} \times (CE^{1/2}\epsilon^{1-1/3})^2$ . Hence the size of the set leading to such recollisions goes to zero.

If  $i$  has at least one reflection, consider  $\omega_i^l$  its last reflection before the recollision,  $\tau_l$  the time of the last reflection and  $\tau$  the time of recollision. In order to have a recollision, there exists a direction  $\nu_{\text{rec}}$  such that

$$(x_j(\tau) - x_i(\tau)) + (\tau - \tau_l)(v_j - |v_i(t_k^-)|\omega_i^l) = \epsilon \nu_{\text{rec}}$$

$$\Rightarrow \omega_i^l = \frac{v_j}{|v_i(t_k^-)|} + \frac{1}{|v_i(t_k^-)|(\tau - \tau_l)} ((x_i(\tau) - x_j(\tau)) + \epsilon \nu_{\text{rec}}).$$

Thus  $\omega_i^l$  is in  $(v_j/|v_i(t_k^-)|) + \mathfrak{C}[(x_i(\tau) - x_j(\tau)), 2 \arcsin \epsilon/|x_i(\tau) - x_j(\tau)|] \cap \mathbb{S}^2$ . Because the norm of the velocities lays between  $\epsilon^{1/4}$  and  $E^{1/2}$ , for  $d := |x_i(\tau) - x_j(\tau)|$ ,  $\omega_i^l$  stays in the intersection of



$\mathbb{S}^2$  and a cylinder of radius

$$\left(1 + \frac{|v_j|}{|v_i(t_k^-)|}\right) 2 \tan\left(\arcsin \frac{\epsilon}{|x_i(\tau_l) - x_j(\tau_l)|}\right) = CE^{1/2} \frac{\epsilon^{1-1/4}/d}{\sqrt{1 - \epsilon^2/d^2}}.$$

Finally the set of bad directions of reflection  $\omega_i^l$  is of size at most :

$$(44) \quad CE^{1/4} \frac{(\epsilon^{1-1/4}/d)^{1/2}}{(1 - \epsilon^2/d^2)^{1/4}} \leq CE^{1/4} \frac{(\epsilon^{3/4}/d)^{1/2}}{(1 - \epsilon^2/(T^2E))^{1/4}} \leq CE^{1/2} (\epsilon^{3/4}/d)^{1/2}$$

using that a particle can cross at most a distance  $TE^{1/2}$  and that  $\epsilon$  is small enough. In order to control the size of the bad set, we have to cut the trajectories such that  $d < \epsilon^{1/2}$ . Let  $\{x_j^\parallel\}$  the projection of the straight lines  $\{x_j(t), t \in \mathbb{R}\}$  on  $\partial\tilde{\Lambda}$ . The last change of direction of the particle  $i$  is at the previous reflection  $\omega_i^{l-1}$  or at the creation  $k$ , and occurs at point  $X_i$ . Because the distance between  $X_i$  and  $\partial\Lambda$  is greater than  $\epsilon^{1/3}$ , the particle  $i$  reaches the boundary at distance less than  $\epsilon^{1/2}$  of  $\{x_j^\parallel\}$  only if its velocity  $v_i(\tau_l^+)$  forms an angle less than  $C\epsilon^{1/2-1/3}$  with the plane passing by  $\{x_j^\parallel\}$  and  $X_i$ . Integrating on the  $\omega_i^{l-1}$  or  $(\nu_k, \bar{\nu}_k)$  according to the nature of the previous change of direction, the size of parameters such that  $d < \epsilon^{1/2}$  goes to zero.

**Now we treat the case where  $i$  and  $j$  are both involved in the creation  $k$  (say  $k = i$ ).** We begin by cutting the time where  $x_j(t_i)$  is close to the boundary. Because  $|v_j(t_i^+) \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$ , for  $t_i$  outside a set of size  $CE^{1/2}T\epsilon^{1/3-1/4}$ ,  $x_j(t_i)$  is at distance greater than  $(\epsilon^{1/3} + \epsilon)$  of the boundary.

In the case where neither  $i$  nor  $j$  have a reflection, the particles do not see the boundary and we can treat it as in the case where the domain is  $\mathbb{T}^3$ .

Note that in the precedent paragraph, if  $i$  had two reflections, we parametrized the bad set by the two last directions of reflection. Thus the same reasoning works and the size of parameters leading to such recollisions goes to zero.

We have finally to deal with two cases: if the two particles have a reflection or if only one does.

We treat first the case where only  $i$  has a reflection which occurs at time  $\tau_l$ . Because  $x_i(t_i^-)$  is at distance at least  $\epsilon^{1/3}$  of  $\partial\Lambda$  and  $|v_i(t_i^-) \cdot \bar{\mathbf{e}}_1| < E^{1/2}$ ,  $(t_i - \tau_l)$  is greater than  $\epsilon^{1/3}E^{-1/2}$ . The distance  $d$  between  $x_i(\tau_l)$  and  $x_j(\tau_l)$  is greater than

$$d \geq |[\epsilon\nu_i + (\tau_l - t_i)(v_i(t_i^-) - v_j(t_i^-))] \cdot \bar{\mathbf{e}}_1| > E^{-1/2}\epsilon^{1/3}\epsilon^{1/4} - \epsilon > \epsilon^{1/2},$$

using that  $|(v_i(t_i^-) - v_j(t_i^-)) \cdot \bar{\mathbf{e}}_1| > \epsilon^{1/4}$ . Thus we can use the estimation 44. The case where  $j$  has a reflection can be treated similarly.

In the case where the two particles have a reflection, we suppose that the last reflection involves the particle  $i$ . In the covering domain  $\tilde{\Lambda}$ , we apply the formula 44 with

$$d = |\vec{d}_0 + (\tau_{\bar{l}} - \tau_l)|v_j|\omega_j^{\bar{l}}|$$

where  $\tau_l$  (respectively  $\tau_{\bar{l}}$ ) is the time of the last reflection of the particle  $i$  (respectively  $j$ ),  $\omega_j^{\bar{l}}$  is the last direction of reflection, and  $\vec{d}_0 := x_j(\tau_{\bar{l}}) - x_i(\tau_l)$ . We can use the two orthogonal decompositions :

$$\begin{aligned} \vec{d}_0 + (\tau_{\bar{l}} - \tau_l)|v_j|\omega_j^{\bar{l}} &= \left(\vec{d}_0 \cdot \omega_j^{\bar{l}} + (\tau_{\bar{l}} - \tau_l)|v_j|\right) \omega_j^{\bar{l}} + \vec{d}_0^\perp \\ &= \left(|\vec{d}_0| + (\tau_{\bar{l}} - \tau_l)|v_j|\omega_j^{\bar{l}} \cdot \frac{\vec{d}_0}{|\vec{d}_0|}\right) \frac{\vec{d}_0}{|\vec{d}_0|} + (\tau_{\bar{l}} - \tau_l)|v_j|\omega_j^{\bar{l}\perp}. \end{aligned}$$

Denoting  $\theta$  the angle between  $\vec{d}_0$  and  $\omega_j^{\bar{j}}$ ,

$$d \geq \sup \left( \left| \vec{d}_0^\perp \right|, |\tau_{\bar{l}} - \tau_l| |v_j| \left| \omega_j^{\bar{l}\perp} \right| \right) \geq |\sin \theta| \sup \left( \left| \vec{d}_0 \right|, |\tau_{\bar{l}} - \tau_l| |v_j| \right),$$

and the size of bad directions  $\omega_i^{\bar{l}}$  is at most

$$\frac{CE^{1/4} (\epsilon^{1-1/4})^{1/2}}{|\sin \theta|^{1/2}} \inf \left( \left| \vec{d}_0 \right|^{-1/2}, (|\tau_{\bar{l}} - \tau_l| |v_j|)^{-1/2} \right).$$

The term  $|\sin \theta|^{-1/2}$  is integrable with respect to the measure  $d\omega_j^{\bar{j}}$ . Hence the set of bad directions  $(\omega_i^{\bar{l}}, \omega_j^{\bar{l}})$  is of size at most

$$CE^{1/4} \epsilon^{3/8} \inf \left( \left| \vec{d}_0 \right|^{-1/2}, (|\tau_{\bar{l}} - \tau_l| |v_j|)^{-1/2} \right).$$

If  $i$  and  $j$  have a reflection on a different component of  $\partial\Lambda$ ,  $|\vec{d}_0| \geq 1$ . Else, denoting  $h$  (respectively  $h + \delta h$ ) the distance of  $j$  (respectively  $i$ ) from the side of  $\partial\Lambda$  where reflections occur,

$$\begin{aligned} |v_j|(\tau_{\bar{l}} - \tau_l) &= |v_j| \left| \frac{h}{v_j \cdot \vec{e}_1} - \frac{h + \delta h}{v_i \cdot \vec{e}_1} \right| \\ &\geq \frac{h|(v_i - v_j) \cdot \vec{e}_1|}{|v_i \cdot \vec{e}_1|} \frac{|v_j|}{|v_j \cdot \vec{e}_1|} - \frac{|\delta h|}{|v_j \cdot \vec{e}_1|} \\ &\geq E^{1/2} \epsilon^{1/3+1/4} - \epsilon^{1-1/4} \\ &\geq \epsilon^{7/12} \end{aligned}$$

for  $E$  large enough. Hence the set of the bad parameters is of size

$$C(E, R, T) \epsilon^{3/8-7/24} = C(E, R, T) \epsilon^{1/12}.$$

This allows us to conclude that  $|\mathcal{P}_2|$  converges to 0. One can estimate the set of overlaps  $\mathcal{P}'_2$  and  $\mathcal{P}_3$  in the same way. This concludes the proof of Theorem 5.1.  $\square$

#### APPENDIX A. CARLEMAN'S PARAMETRIZATION AND SCATTERING ESTIMATES

In section 6 we need to estimate some singular integrals with respect to the measure  $|(v - v_*) \cdot \nu| dv dv_*$  where  $v_i$  can represent  $v_*$ ,  $v'$  or  $v'_*$ . In this appendix we give the detailed statement and proofs.

We start by recalling the Carleman's collision parameters

$$(45) \quad \begin{cases} \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow C := \{(v', v'_*) \in \mathbb{R}^3 \times \mathbb{R}^3, (v' - v) \cdot (v'_* - v) = 0\} \\ (v_*, \nu) \mapsto (v', v'_*) \end{cases}$$

which map the measure  $|(v - v_*) \cdot \nu| dv_* d\nu$  into  $dv' dS(v'_*)$  where  $(v', v'_*)$  are given by the scattering and  $dS$  is the Lebesgue measure on the affine plane passing through  $v$  and normal to  $(v' - v)$ .

We can then prove our first estimation lemma:

**Lemma A.1.** *Fix  $v \in \mathbb{R}^3$  and  $a < b$  two real numbers. Then for  $v_i$  equal to  $v_*$ ,  $v'$  or  $v'_*$ , for some constant  $C > 0$*

$$(46) \quad \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbb{1}_{(v_i \cdot \vec{e}_1) \in [a, b]} \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \leq CE^2 |b - a|.$$

*Proof.* In the case where  $v_i = v_*$ , the proof is straightforward. If  $v_i = v'$  we apply Carleman's change of variables and the proof is also direct.

In the case where  $v_i = v'_*$ , we apply Carleman's change of variables. If  $v'_* \cdot \vec{e}_1$  is between  $a$  and  $b$ , then  $v'_*$  stays in a rectangle of size  $E^{1/2} \times |b - a| |\sin(\vec{e}_1, v' - v)|$ . Then integrating with respect to  $v'$ , we obtain the expected bound.  $\square$

From this lemma we can deduce the following estimates:

**Proposition A.2.** *Fix  $v \in \mathbb{R}^3$ . Then for  $v_i$  equal to  $v_*$ ,  $v'$  or  $v'_*$ , for some constant  $C > 0$*

$$(47) \quad \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left( \frac{\epsilon}{|v_i \cdot \vec{e}_1|} \wedge 1 \right) \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \leq CE^2 \epsilon |\log \epsilon|$$

$$(48) \quad \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left( \frac{\epsilon}{|v_i \cdot \vec{e}_1|^2} \wedge 1 \right) \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \leq CE^2 \epsilon^{1/2}$$

*Proof.* We use the previous lemma to decompose the set:

$$\begin{aligned} & \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left( \frac{\epsilon}{|v_i \cdot \vec{e}_1|} \wedge 1 \right) \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \leq \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbb{1}_{|v_i \cdot \vec{e}_1| \in [0, \epsilon]} \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \quad + \sum_{1 \leq n \leq E^{1/2}/\epsilon} \frac{1}{n} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbb{1}_{|v_i \cdot \vec{e}_1| \in [n\epsilon, (n+1)\epsilon]} \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \leq CE^2 \epsilon \left( 1 + \sum_{1 \leq n \leq E^{1/2}/\epsilon} \frac{1}{n} \right) \\ & \leq CE^2 \epsilon |\log E^{1/2}/\epsilon| \leq CE^2 \epsilon |\log \epsilon| \end{aligned}$$

The proof of the second line is similar.

$$\begin{aligned} & \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left( \frac{\epsilon}{|v_i \cdot \vec{e}_1|^2} \wedge 1 \right) \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \leq \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbb{1}_{|v_i \cdot \vec{e}_1| \in [0, \epsilon^{1/2}]} \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \quad + \sum_{1 \leq n} \frac{1}{n^2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbb{1}_{|v_i \cdot \vec{e}_1| \in [n\epsilon^{1/2}, (n+1)\epsilon^{1/2}]} \mathbb{1}_{|v|^2 + |v_*|^2 \leq E} |(v - v_*) \cdot \nu| dv_* d\nu \\ & \leq CE^2 \epsilon^{1/2} \left( 1 + \sum_{1 \leq n} \frac{1}{n^2} \right) \\ & \leq CE^2 \epsilon^{1/2} \end{aligned}$$

$\square$

## APPENDIX B. HYDRODYNAMIC LIMIT

Even if we are actually far from extending our main result to hydrodynamic time scales, we quickly discuss, for the sake of completeness, the link between our kinetic problem and hydrodynamic equations.

We look at the long time behavior (taking a time coordinate  $\tau = t/\alpha$ ) of the system when the Knudsen number converges to zero,  $\alpha := N^{-1}\epsilon^{-2} \rightarrow 0$ . At fixed  $\alpha$  one considers

$$(49) \quad \begin{cases} \alpha \partial_\tau f_\alpha + v \cdot \nabla_x f_\alpha = \frac{1}{\alpha} Q(f_\alpha, f_\alpha) \text{ on } \Lambda \times \mathbb{R}^3 \\ f_\alpha(\tau, x, v) = \int_{\mathbb{S}^2} f_\alpha(\tau, x, |v|\omega) c_3(\omega \cdot n(x))_- d\omega \text{ on } \Sigma^+, \\ f_\alpha(\tau = 0) = f^0. \end{cases}$$

where  $\Sigma^+ := \{(x, v) \in \partial\Lambda \times \mathbb{R}^3, v \cdot n(x) > 0\}$ . We consider the linearisation of the Boltzmann equation for solutions of the form  $f(t, x, v) = M(v)(1 + \delta g(t, x, v))$  when  $\delta \ll 1$ :

$$(50) \quad \begin{cases} \alpha \partial_\tau g_\alpha + v \cdot \nabla_x g_\alpha = \frac{1}{\alpha} \mathcal{L}g_\alpha \text{ on } \Lambda \times \mathbb{R}^3 \\ g_\alpha(\tau, x, v) = \int_{\mathbb{S}^2} g_\alpha(\tau, x, |v|\omega) c_3(\omega \cdot n(x))_- d\omega \text{ on } \Sigma^+ \\ g_\alpha(\tau = 0) = g^0 \end{cases}$$

where  $\mathcal{L}$  is the linearized collision operator around the Gaussian density  $M(v) := \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}|v|^2}$ ,

$$(51) \quad \begin{aligned} \mathcal{L}g(v) &:= \frac{1}{M(v)} (Q(Mg, M) + Q(M, Mg)) \\ &= \int_{\mathbb{S}^2 \times \mathbb{R}^3} (g(v') + g(v'_*) - g(v) - g(v_*)) ((v - v_*) \cdot \nu)_+ M(v_*) dv dv_* \end{aligned}$$

and where  $(v', v'_*)$  are given by the scattering of  $(v, v_*)$ . The operator  $\mathcal{L}$  is self-adjoint and non-positive on  $L^2(M(v)dx dv)$ .

The hydrodynamic limit of this system is given by the following theorem (see [22], or [2] for a proof with a different boundary condition):

**Theorem B.1.** *Let  $g_0 \in L^2(M(v)dx dv)$ . Then the following statements hold:*

- for each  $\alpha > 0$ ,  $T > 0$  there exists a unique solution  $g_\alpha(\tau) \in \mathcal{C}([0, T], L^2(M(v)dv dx))$  of 50,
- the  $g_\alpha$  converge weakly in  $L^2([0, T], L^2(M(v)dv dx))$  to

$$g_{\text{lim}}(\tau, x, v) = \rho(\tau, x) + u(\tau, x) \cdot v + \theta(\tau, x) \frac{|v|^2 - 3}{2}$$

where  $(\rho, u, \theta)$  are the solution of the Stokes-Fourier system:

$$(52) \quad \begin{cases} \partial_\tau u - \nu \Delta u = 0, \quad \partial_\tau \theta - \kappa \Delta \theta = 0 \\ \nabla \cdot u = 0, \quad \nabla(\rho + \theta) = 0 \text{ (incompressible and Boussinesq conditions)} \\ u_{\partial\Lambda} = 0, \quad n \cdot \nabla \theta_{\partial\Lambda} = 0 \\ (\rho, u, \theta)(\tau = 0) = \mathbb{P} \left( \int g^0 M(v) dv, \int g^0 v M(v) dv, \int g^0 \frac{|v|^2 - 3}{2} M(v) dv \right) \end{cases}$$

with  $\mathbb{P}$  is the orthogonal projection on

$$\left\{ (\rho, u, \theta) \in (L^2(\Lambda))^5, \left\{ \begin{array}{l} \nabla \cdot u = 0 \\ \nabla(\rho + \theta) = 0 \end{array} \right. \text{ in the sense of distributions} \right\} \subset (L^2(\Lambda))^5$$

and  $\nu, \kappa$  are two positive constants depending only on the dimension of the domain.

In the theorem,  $\rho$  represents the variation of density of the system,  $u$  the mean velocity and  $\theta$  the variation of temperature. We observe that at the boundary  $n(x) \cdot \nabla \theta = 0$ , thus no transfer of energy is present.

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### REFERENCES

- [1] R. K. Alexander, *The infinite hard-sphere system*, ProQuest LLC, Ann Arbor, MI, 1975, URL [http://gateway.proquest.com/openurl?url\\_ver=Z39.88-2004&rft\\_val\\_fmt=info:ofi/fmt:kev:mtx:dissertation&res\\_dat=xri:pqdiss&rft\\_dat=xri:pqdiss:7615092](http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:7615092), Thesis (Ph.D.)—University of California, Berkeley.
- [2] T. Bodineau, I. Gallagher and L. Saint-Raymond, A microscopic view of the Fourier law, *Comptes Rendus Physique*, **20** (2019), 402–418.
- [3] L. Boltzmann, *Lectures on gas theory*, Translated by Stephen G. Brush, University of California Press, Berkeley-Los Angeles, Calif., 1964.
- [4] S. Caprino and M. Pulvirenti, The Boltzmann-Grad limit for a one-dimensional Boltzmann equation in a stationary state, *Commun. Math. Phys.*, **177** (1996), 63–81.
- [5] N. Catapano, *Stime  $L^\infty$  per il flusso di Knudsen con condizioni diffusive al bordo*, Master’s thesis.
- [6] C. Cercignani, *Rarefied gas dynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000, From basic concepts to actual calculations.
- [7] C. Cercignani, R. Illner and M. Pulvirenti, *The mathematical theory of dilute gases*, vol. 106 of Applied Mathematical Sciences, Springer-Verlag, New York, 1994, URL <https://doi.org/10.1007/978-1-4419-8524-8>.
- [8] R. Denlinger, The propagation of chaos for a rarefied gas of hard spheres in the whole space, *Arch. Ration. Mech. Anal.*, **229** (2018), 885–952.
- [9] T. Dolmaire, *Etude mathématique de la dérivation de l’équation de Boltzmann dans un domaine à bord*, PhD thesis, Université de Paris, 2019.
- [10] T. Dolmaire, About Lanford’s theorem in the half-space with specular reflection, 2021, ArXiv:2102.05513.
- [11] R. Esposito, Y. Guo, C. Kim and R. Marra, Non-isothermal boundary in the Boltzmann theory and Fourier law, *Commun. Math. Phys.*, **323** (2013), 177–239.
- [12] R. Esposito and R. Marra, Stationary non equilibrium states in kinetic theory, *J. Stat. Phys.*, **180** (2020), 773–809.
- [13] R. Esposito, Y. Guo, C. Kim and R. Marra, Stationary solutions to the Boltzmann equation in the hydrodynamic limit, *Ann. PDE*, **4** (2018), 119, Id/No 1.
- [14] I. Gallagher, L. Saint-Raymond and B. Texier, *From Newton to Boltzmann: hard spheres and short-range potentials*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2013.
- [15] V. Gerasimenko and I. Gapyak, Low-density asymptotic behavior of observables of hard sphere fluids, *Adv. Math. Phys.*, **2018** (2018), 11, Id/No 6252919.
- [16] S. Goldstein, J. L. Lebowitz and E. Presutti, Mechanical system with stochastic boundaries, Random fields. Rigorous results in statistical mechanics and quantum field theory, Esztergom 1979, Colloq. Math. Soc. Janos Bolyai 27, 403-419 (1981)., 1981.
- [17] H. Grad, On the kinetic theory of rarefied gases, *Comm. Pure Appl. Math.*, **2** (1949), 331–407, URL <https://doi-org.acces.bibliotheque-diderot.fr/10.1002/cpa.3160020403>.
- [18] J. P. Guiraud, Problème aux limites intérieur pour l’équation de Boltzmann linéaire, *J. Méc., Paris*, **9** (1970), 443–490.
- [19] J.-P. Guiraud, Problème aux limites interieur pour l’équation de Boltzmann en regime stationnaire, faiblement non linéaire, *J. Méc., Paris*, **11** (1972), 183–231.
- [20] R. Illner and M. Pulvirenti, Global validity of the Boltzmann equation for two- and three-dimensional rare gas in vacuum. Erratum and improved result: “Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum” [Comm. Math. Phys. **105** (1986), no. 2, 189–203; MR0849204 (88d:82061)] and “Global validity of the Boltzmann equation for a three-dimensional rare gas in vacuum” [ibid. **113** (1987), no. 1, 79–85;

- MR0918406 (89b:82052)] by Pulvirenti, *Comm. Math. Phys.*, **121** (1989), 143–146, URL <http://projecteuclid.org/euclid.cmp/1104178007>.
- [21] O. E. Lanford III, Time evolution of large classical systems, in *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)*, 1975, 1–111. Lecture Notes in Phys., Vol. 38.
  - [22] C. Le Bihan, *Convergence d'un système de sphères dures vers une solution de l'équation de Stokes-Fourier avec bord*, Master's thesis, ENS de Lyon, 2019.
  - [23] C. D. Levermore, Mathematics of kinetic theory, 2012, Lecture notes, <https://terpconnect.umd.edu/~lvrmr/2012-2013-F/Classes/AMSC698/NOTES/Lec06.pdf>.
  - [24] M. Pulvirenti, C. Saffirio and S. Simonella, On the validity of the Boltzmann equation for short range potentials, *Rev. Math. Phys.*, **26** (2014), 1450001, 64, URL <https://doi-org.acces.bibliotheque-diderot.fr/10.1142/S0129055X14500019>.
  - [25] M. Pulvirenti and S. Simonella, On the evolution of the empirical measure for the hard-sphere dynamics, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **10** (2015), 171–204.
  - [26] H. Spohn, *Large scale dynamics of interacting particles*, Berlin: Springer-Verlag, 1991.