Annales de la faculté des sciences de Toulouse Volume I, n° 0, -1
pp. 000-000

# On essential-selfadjointness of differential operators on closed manifolds [(*)]

Yves Colin de Verdière [(1)] and Corentin Le Bihan [(2)]

**Abstract.** — The goal of this paper is to present some arguments leading to the following conjecture: a formally self-adjoint differential operator on a closed manifold is essentially self-adjoint if and only if the Hamiltonian flow of its symbol is complete. This holds for differential operators of degree two on the circle, for differential operators of degree one on any closed manifold and for Lorentzian Laplacians on generic Lorentzian surfaces.

**Résumé.** — Le but de cet article est de présenter des arguments conduisant à la conjecture suivante : sur une variété compacte, un opérateur pseudo-différentiel formellement symétrique est essentiellement auto-adjoint si et seulement si le flot Hamiltonien du symbole est complet. Nous montrons cette conjecture pour les opérateurs différentiels de degré 2 dans le cas du cercle, pour les opérateurs différentiels de degré 1 et pour le laplacien lorentzien des surfaces Lorentziennes génériques.

## 1. Introduction

Let $X$ be a compact smooth[(1)] manifold without boundary equipped with a smooth density $|\mathrm{d}x|$. We will often denote by $L^2$ the Hilbert space $L^2(X, |\mathrm{d}x|)$. Let $P$ be a differential operator of degree 2 on $X$ with smooth coefficients acting on complex valued functions. We assume in what follows that $P$ is symmetric, i.e., for any pair of smooth complex valued functions $f, g$ on $X$, we have $\int_X Pf\ \bar{g}|\mathrm{d}x| = \int_X f\ \overline{Pg}|\mathrm{d}x|$.

[(*)] Reçu le 20 avril 2020, accepté le 9 octobre 2020.

[(1)] Institut Fourier, Université Grenoble-Alpes, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402 Saint Martin d'Hères Cedex, France — yves.colin-de-verdiere@univ-grenoble-alpes.fr

[(2)] UMPA, UMR 5669 CNRS, ENS de Lyon Site Monod, 46 Allée d'Italie, 69364 Lyon Cedex 07, France — corentin.le-bihan@ens-lyon.fr

Article proposé par Gilles Carron.

[(1)] In all this paper, "smooth" means $C^\infty$

The adjoint $P^\star$ of $P$ is then defined as follows: the domain $D(P^\star)$ is the set of distributions $f \in L^2$ so that $Pf$ belongs to $L^2$ and $P^\star$ is the operator $P$ acting on such distributions.

Definition 1.1. — *A symmetric linear differential operator with smooth coefficients $P$ is* essentially self-adjoint *(denoted ESA in what follows) if the graph of $P^\star$ in $L^2 \times L^2$ is the closure of the graph of $P$ with domain the smooth functions on $X$. More explicitely, for each $v = Pu$ with $u, v \in L^2$, there exists a sequence $(u_n, v_n)$ with $u_n$ smooth, $v_n = Pu_n$ and $(u_n, v_n) \to (u, v)$ in $L^2 \times L^2$.*

The ESA property is also called *Quantum completeness* because the evolution equation $\mathrm{d}u/\mathrm{d}t = iPu$ with $u(t = 0) = f$, with $f$ smooth, has then an unique solution defined for $t \in \mathbb{R}$ and denoted $u(t) = \exp(itP)f$ (see [12, §VIII]).

On the other hand, $P$ admits a principal symbol and also a sub-principal symbol: if one chooses local coordinates $x = (x_1, \dots, x_n)$ so that $|\mathrm{d}x| = |\mathrm{d}x_1 \dots \mathrm{d}x_n|$, and if $P = \sum \frac{\partial}{\partial x_k} a_{kl}(x) \frac{\partial}{\partial x_l} + \sum b_k(x) \frac{\partial}{\partial x_k} + c(x)$, the principal symbol is $p_2 := -\sum a_{kl}(x)\xi_k\xi_l$ and the sub-principal symbol is $p_1 := i\sum b_k(x)\xi_k$ (see [7, §3]). Note that, if $P$ is symmetric, $p_1$ and $p_2$ are real valued. We denote $p = p_2 + p_1$ and call it the symbol of $P$. Note that $p$ is real valued if $P$ is formally symmetric. The symbol $p$ is independent of the choice of local coordinates as soon as interpreted as a function on the cotangent space $T^\star X$. The cotangent space is a symplectic manifold and one can use $p$ as an Hamiltonian function on it. We say that $P$ is *classically complete* if the Hamiltonian flow of $p$ is complete: it means that the maximal interval of definition of any integral curve of the Hamiltonian vector field of $p$ is $\mathbb{R}$.

*A natural question is then: how are classical and quantum completeness related?* The goal of this article is to give very partial answers to this question. We will state a possible answer as the

Conjecture 1.2. — *Let $P$ be a formally self-adjoint differential operator of degree 2 on $C^\infty(X)$ where $X$ is a closed smooth manifold equipped with a smooth density $|\mathrm{d}x|$, classical and quantum completeness are equivalent.*

As we will see, this conjecture holds true in the following cases:

(1) Differential operators of degree 2 on the circle of the form
$$P = a(x)\mathrm{d}_x^2 + \dots$$
where all zeroes of $a$ are of finite order.
(2) Differential operators of degree 1.
(3) Generic and conformally flat Lorentzian Laplacians on 2-tori.

Let us describe other known results on this question: a classical result is that classical completeness and quantum completeness are not equivalent in general; examples of Schrödinger operators on $\mathbb{R}$ are given in [12, §X-1, p. 155–157]. However the potentials involved are quite complicated near infinity: they do not admit a polynomial asymptotic behaviour. A classical results in this domain is Gaffney's Theorem [6] which states that, if a Riemannian manifold $(X, g)$ is complete, the Laplace operator on it is ESA. For a clear proof, see [4, p. 151–152]. For a more recent work on this aspect, see [1].

#### Acknowledgements

Many thanks to Christophe Bavard, Yves Carrière, Etienne Ghys, Nicolas Lerner and Bernard Malgrange for several discussions helping us. Many thanks also to the referee for many remarks helping us to make the paper more accessible.

## 2. General facts on ESA operators

### 2.1. Abstract context

Let us recall some classical results which can be found in [12, §X]. Let $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ be an Hilbert space. A linear operator $P : D(P) \to \mathcal{H}$ with $D(P)$ a dense subspace of $\mathcal{H}$ is said to be symmetric if, for all $x, y \in D(P)$, $\langle Px|y \rangle = \langle x|Py \rangle$. The closure of $P$ is the operator $\overline{P}$ whose graph is the closure of the graph of $P$ in $\mathcal{H} \times \mathcal{H}$. The adjoint $P^\star$ of $P$ is defined as follows: the domain $D(P^\star)$ is the set of $x \in \mathcal{H}$ so that $y \to \langle Py|x \rangle$ as defined on $D(P)$ extends continously to $\mathcal{H}$. We have then $\langle Py|x \rangle \equiv \langle y|z \rangle$ and we define $P^\star x = z$. The operator $P$ is ESA if $P^\star$ is the closure of $P$. In other words, $P$ has an unique self-adjoint extension. A useful property is the following one

Theorem 2.1. — *A symmetric operator $P$ on an Hilbert space $\mathcal{H}$ is ESA if and only if the spaces $\ker_{\mathcal{H}}(P^\star \pm i)$ are $\{0\}$.*

### 2.2. The case of differential operators on compact manifolds without boundary

Recall that a differential operator with smooth coefficients acts on Schwartz distributions and in particular on $L^2$ functions. In particular, we see that any symmetric elliptic operator $P$ on a closed manifold is ESA: if $(P \pm i)u = 0$, $u$ is smooth and the result follows from the symmetry of $P$. This is why we are only interested here to non elliptic operators.

### 2.3. The case of differential operators on a compact interval

Let $X := [\alpha, \beta]$ be a compact interval. We consider a differential operator $P$ of degree 2 whose coefficients are smooth up to the boundary. Assuming that $P$ is symmetric on $C_c^\infty(]\alpha, \beta[, |dx|)$ (it is usually called formally symmetric), then $P$ is given by the equation (4.1).

We want to describe the *Dirichlet boundary conditions*. For that, we assume that $P$ is elliptic near the boundary, i.e. that $a$ does not vanish at the points of $\partial X$. We will take for domain of $P$ the space $D(P) := C^\infty(\overline{X}, \mathbb{C}) \cap \{f | f|_{\partial X} = 0\}$.

Lemma 2.2. — *The domain of $P^\star$ is then the set of $L^2$ functions $g$ so that $Pg \in L^2$, where $P$ is acting on distributions defined in the interior of $X$ and $g$ vanishes on the boundary.*

*Proof.* — We get first that $Pg \in L^2$ by looking at $\int_X Pf\bar{g}|dx|$ with $f \in C_c^\infty(]\alpha, \beta[)$. It follows that $g$ is continuous near the boundary. Then we have, if $f \in D(P)$,

$$\int_X (Pf\bar{g} - f\overline{Pg})|dx| = [af'g]_\alpha^\beta$$

We have to control the righthandside in terms of the $L^2$ norm of $f$ which is clearly not possible if $g$ does not vanish on the boundary because $a$ does not vanish at on $\partial X$. □

### 2.4. Localization

Let us prove the following localization

Lemma 2.3. — *Let $P$ be a symmetric operator of degree 2 on a the circle. Let $Z \subset X$ be the closed set of points where $P$ is not elliptic. We assume that $Z$ is a finite set $Z = \{x_1, \ldots, x_N\}$. Let $\Omega = \bigcup_{j=1}^N [\alpha_j, \beta_j]$ be a neighbourhood of $Z$ so that $P$ is elliptic near the boundary of $\Omega$. Then $P$ is ESA if and only if the Dirichlet restriction $P_\Omega$ of $P$ to $\Omega$ is ESA.*

*Proof.* — Let us first prove that, if $P_\Omega$ is ESA, $P$ is ESA: let us take a $\rho \in C_c^\infty(\Omega)$ with $\rho \equiv 1$ near $Z$. Then, if $Pu = v$ with $u, v \in L^2(X)$, $(1 - \rho)u$ belongs to the Sobolev space $H^2(X)$ by ellipticity of $P$ on the support of $1-\rho$. In particular $P((1-\rho)u) \in L^2$. There exists $(u_n', v_n' = Pu_n')$ a sequence of smooth functions converging to $((1-\rho)u, P((1-\rho)u)$ in $L^2$ by density of $C^\infty(X)$ in $H^2(X)$. We have now $P(\rho u) = w$ with $\rho u, w \in L^2$ and support$(\rho u) \subset \Omega$. ESA of $P_\Omega$ allows to approximate $(\rho u, w)$ by smooth functions $(u_n'', v_n'' = Pu_n'')$ and we can assume that $u_n''$ vanishes near the boundary because $\rho u$ does. Then $(u_n' + u_n'', v_n' + v_n'')$ are smooth, converge to $(u, v)$ in $L^2 \times L^2$ and $P(u_n' + u_n'') = v_n' + v_n''$. This allows to conclude that $P$ is ESA.

Let us now prove that if $P$ is ESA, $P_\Omega$ is ESA: let us start with $P_\Omega u = v$ with $(u, v) \in L^2(\Omega)$, $u \in H^2$ near the boundary and $u(\partial\Omega) = 0$. We choose $\rho \in C_c^\infty(\Omega)$ with $\rho = 1$ near $Z$. Similarly to the previous argument, we decompose $u = \rho u + (1-\rho)u$. And by ellipticity of $P$ near $\partial\Omega$ we get that $(1-\rho)u$ belongs to the Sobolev space $H_0^2(\Omega)$ of distributions which are in $H^2(\Omega)$ and vanish at the boundary. By density of smooth functions vanishing at the boundary in the Sobolev space $H_0^2$, we get an approximating sequence to $((1-\rho)u, P((1-\rho)u)$. Now we are left with $\rho u$ with $P(\rho u) \in L^2$ and we use the fact that $P$ is ESA to get an approximating sequence $(u_n', Pu_n')$ on $X$. Choosing $\rho_1 \in C_c^\infty(\Omega)$ with $\rho_1 = 1$ on the support of $\rho$, we take $u_n'' = \rho_1 u_n'$ and we get an approximating sequence for $(\rho u, P(\rho u)$. This allows to conclude. □

Note that the previous localization result extends probably to higher dimensional manifolds, but this extension is much less simple.

## 3. Essential self-adjointness of differential operators of degree 1

The following property goes back to Friedrichs [5] as cited by Hörmander [8].

Lemma 3.1. — *Let $P$ be a differential operator of degree $1$ on a closed manifold $X$ and $u \in L^2(X)$ so that $Pu \in L^2(X)$. There exists a sequence $u_j \in C^\infty(X)$ so that $u_j \to u$ and $Pu_j \to Pu$ both in $L^2(X)$.*

If $P$ is symmetric, this implies that the closure in $L^2 \oplus L^2$ of the graph of $P$ restricted to smooth functions is the graph of the adjoint of $P$. Hence $P$ with domain $C^\infty(X)$ is essentially self-adjoint.

Theorem 3.2. — *Any symmetric differential operator of degree $1$ on a closed manifold is essentially self-adjoint.*

This holds in particular for differential operators of the form $P := i(V + \frac{1}{2}\operatorname{div}_{|\mathrm{d}x|}(V))$ where $V$ is a vector field and $\operatorname{div}_{|\mathrm{d}x|}(V) := d(\iota(V)\mathrm{d}x)/\mathrm{d}x$ where $\iota$ is the inner product.

In the note [9], Nicolas Lerner remarks that this property extends to pseudo-differential operator of degree 1.

This is related to our problem because then the Hamiltonian flow is complete at infinity: the Hamiltonian vector field of $p$ is bounded by $C\|\xi\|$ and the completeness at infinity follows from Gronwall lemma.

## 4. Sturm–Liouville operators on the circle

### 4.1. Main result

Any symmetric operator on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ equipped with the Lebesgue measure $|\mathrm{d}x|$ can be written as

$$P = \mathrm{d}_x a(x)\mathrm{d}_x - ib(x)\mathrm{d}_x - i\frac{1}{2}b'(x) + c(x) \tag{4.1}$$

where $\mathrm{d}_x := \mathrm{d}/\mathrm{d}x$, $a$, $b$, $c$ are smooth real valued periodic functions of period 1. We assume always in what follows that the zeroes of $a$ are of finite multiplicities. The symbol $p$ of $P$ is

$$p = -a(x)\xi^2 + b(x)\xi$$

The term $c(x)$ plays no role in the essential self-adjointeness, so we will forget it in what follows.

Our main result is

Theorem 4.1. — *For operators $P$ of the previous form, classical completeness of the Hamiltonian flow of $p$ is equivalent to quantum completeness of $P$.*

Our proof consists in describing the properties of $a$ and $b$ leading to classical completeness and to study the quantum completeness in the corresponding cases.

Note that the result in the case where the zeroes of $a$ are non degenerate is also proved using some microlocal analysis in [14].

### 4.2. Classical completeness

We have the

Theorem 4.2. — *Let $p := -a(x)\xi^2 + b(x)\xi$ where the zeroes of $a$ are of finite multiplicity. Then the Hamiltonian flow of $p$ is complete on $T^\star S^1$ if and only if the zeroes of $a$ are not simple and $b$ vanishes at these zeroes. Moreover, this flow is complete if and only if it is null complete, i.e. complete when restricted to $p^{-1}(0)$.*

*Proof.* — Recall that the Hamiltonian differential equation writes

$$\mathrm{d}x/\mathrm{d}t = -2a(x)\xi + b(x), \ \mathrm{d}\xi/\mathrm{d}t = a'(x)\xi^2 - b'(x)\xi.$$

The function $p$ is constant along the integral curves. We will denote by $(x_0, \xi_0)$ the data at time 0 of the integral curves that we will consider.

The proof splits into three cases:

*Case 1: Assume that $a(0) = 0$ and $b(0) > 0$.* — We will show that the flow is not null complete. Let us look at the set $p = 0$ near $x = 0$. This set is the union of the disjoint curves $\{\xi = 0\}$ and $C := \{a(x)\xi - b(x) = 0\}$. The curve $C \cap \{x > 0\}$ is oriented by the flow so that $x$ is decaying because then $\mathrm{d}x/\mathrm{d}t = -b(x)$. Let us start on $C$, with $x_0 > 0$ small enough and $\xi_0$ so that $-a(x_0)\xi_0 + b(x_0) = 0$. Then $-a(x(t))\xi(t) + b(x(t)) = 0$ for all $t$. Along $C$, we have $\mathrm{d}x/\mathrm{d}t = -b(x)$. Hence, there exists $t_0 > 0$ so that $x(t_0) = 0$ and $\xi(t_0) = +\infty$. The flow is not null complete: the maximal integral curve is only defined up to $t_0^-$.

*Case 2: Assume that $0$ is a non degenerate zero of $a$ and $b(0) = 0$.* — Let us start with $x_0 = 0$ and $\xi_0 \neq 0$. We have $x(t) = 0$ for all $t$ and $\mathrm{d}\xi/\mathrm{d}t = a'(0)\xi^2 - b'(0)\xi$. The solution of this differential equation is not defined for all $t$'s because $a'(0) \neq 0$. The flow is not null complete.

*Case 3: Assume now that zeroes of $a$ are degenerate and $b$ vanishes on $a^{-1}(0)$.* — We want to prove that the flow is complete. We have, by conservation of $p$, for any integral curve, there exists $E$ so that $-a(x)\xi^2+b(x)\xi \equiv E$. It follows that $\xi$ stays bounded on any compact interval in $x$ disjoint from $a^{-1}(0)$. We need only to consider what happens when $x(t)$ comes close to a zero of $a$, says $x = 0$.

Let us first assume that $x_0 = 0$, then $x(t) \equiv 0$ for all $t$ and $\mathrm{d}\xi/\mathrm{d}t = -b'(0)\xi$. The trajectory is complete.

If $x_0 \neq 0$ is close to 0, we get $\mathrm{d}x/\mathrm{d}t = \pm\sqrt{-4a(x)E - b(x)^2} = O(|x|)$. It follows that $x(t)$ does not reach 0 in finite time and hence the integral curve ie defined for all times. ☐

### 4.3. Simple zeroes of $a$

We will show in this section that if $a(0) = 0$ is a simple zero of $a$ then $P$ is not ESA.

Let $I := [-\alpha, \alpha]$ with no other zeroes of $a$ inside $I$. The point 0 is a regular singular point (see Appendix A) of the differential equation $(P - i)u = 0$.

The indicial equation writes $Ar^2 - iBr = 0$ with $A := a'(0),\ B = b(0)$. Hence the solutions of this equation near 0 writes, for $x > 0$, $y(x) = f(x) + x_+^{iB/A} g(x)$ if $B \neq 0$ and $y(x) = f(x) + g(x)\log x$ if $B = 0$ with $f, g$ smooth up to $x = 0$ (see Appendix A) and similarly for $x < 0$ with $x_-$ and $\log(-x)$.

Let $y_+$ be the unique solution of $(P - i)y_+ = 0,\ y_+'(\alpha) = 0,\ y_+'(\alpha) = 1$ on $]0, \alpha]$, so that $y_+$ satisfies the Dirichlet boundary condition at $\alpha$. And define $y_-$ similarly with $y_-(-\alpha) = 0$. If we extend $y_+$ by zero for $x < 0$, we get a Schwartz distribution $Y_+$ and $(P - i)Y_+$ is supported by the origine. We have $PY_+ = \mathrm{d}_x \mathrm{d}_x a Y_+ + \mathrm{d}_x [a, \mathrm{d}_x] Y_+ - ib\mathrm{d}_x Y_+ - ib' Y_+/2$. We check that $\mathrm{d}_x a Y_+$ is in $L^2_{\text{loc}}$. So that $(P - i)Y_+$ is near 0 in the Sobolev space $H^{-1}$, because $\mathrm{d}_x a Y_+ \in L^2$. The derivatives $\delta'(0), \ldots$ of the Dirac distribution are not in $H^{-1}$. We have the same result for $Y_-$. It follows that $(P - i)Y_\pm = \mu_\pm \delta(0)$.

Hence there is a non zero linear combination $Y$ of $Y_+$ and $Y_-$ which satisfies $(P - i)Y = 0$ and the Dirichlet boundary conditions at $\pm\alpha$. This proves that $P_I$ is not ESA and hence $P$ is not ESA by Lemma 2.3.

### 4.4. Degenerate zeroes where $b(0)$ vanishes

Let us assume that all zeroes of $a$ are degenerate. Then if $I = ]c, d[$ is an interval between two zeroes of $a$ and assume $a > 0$ on $I$, we will show that there is an explicit unitary map from $L^2(I, |\mathrm{d}x|)$ onto $L^2(\mathbb{R}, |\mathrm{d}y|)$ sending $C_c^\infty(I)$ (the set of operator with compact support on $I$) into $C_c^\infty(\mathbb{R})$ (set of operator with compact support on $\mathbb{R}$) and sending $P$ to $Q = \mathrm{d}_y^2 + V$ with an explicit $V$.

*First step: a gauge transform.* — Let us consider $P_S := e^{-iS} P e^{iS}$ where $S$ is smooth and real valued. We get, by an easy calculation,

$$P_S = \mathrm{d}_x a \mathrm{d}_x - i(b - 2aS')\mathrm{d}_x - i(b'/2 - a'S' - aS'') - aS'^2 + bS'$$

Choosing $S$ so that $S' = b/2a$, we get

$$P_S = \mathrm{d}_x a \mathrm{d}_x + b^2/4a$$

*Second step: a change of variable.* — Let us choose $x_0 \in I$. Let us define $y = \phi(x) = \int_{x_0}^{x} a^{-\frac{1}{2}}(t)\mathrm{d}t$. The map $\phi$ is smooth diffeomorphism in $I$ onto $\mathbb{R}$. Let us introduce the unitary transform $\Omega : L^2(I, |\mathrm{d}x|) \to L^2(\mathbb{R}, |\mathrm{d}y|)$ defined by $\Omega f(\phi(x)) = a(x)^{1/4} f(x)$. We compute $P_\Omega := \Omega P_S \Omega^\star$. We get $P_\Omega = \mathrm{d}_y^2 + V(y)$ with

$$V(y) = \left( \frac{b^2}{2a} + \frac{a'^2}{16a} - \frac{a''}{4} \right) (\phi^{-1}(y))$$

$(a'^2/16a + a''/4)$ is bounded. The derivative of $(b/a^{1/2}) \circ \phi^{-1}$ is

$$\left( b' - \frac{a'b}{2a} \right) \circ \phi^{-1}$$

which is also bounded. So $(b/a^{1/2}) \circ \phi^{-1}$ is bounded by $C(1 + |y|)$ and we get that $V(y) \leqslant C(y^2 + 1)$.

It follows then from the Farine–Lavis Theorem ([12, Thm. X.38]) that $P_\Omega$ is ESA and hence $P$ is ESA on $C_c^\infty(I)$. It follows that $P$ is ESA on $C_c^\infty(S^1 \setminus a^{-1}(0))$ and a fortiori on $C^\infty(S^1)$.

### 4.5. Degenerate zeroes where $b(0)$ does not vanish

Finally, we study the case where all zeroes of $a$ are degenerate and $b$ does not vanish at least at one of these, say $x = 0$. We will need the following

Lemma 4.3. — *Let us choose a smooth function $E$ on $I := ]0, c]$ so that $E' = b/a$. There exists two independent solutions $u_1$ and $u_2$ of $(P - i)u = 0$ on $I$ such that $u_1$ is smooth up to 0, $u_2 = u_3 e^{iE}$ with $u_3$ smooth up to 0.*

It follows that the functions $a(x)\mathrm{d}_x u_j$ are in $L^2$ and that $P$ is not ESA by the same argument than in Section 4.3.

*Proof.* — We check first the existence of $u_1$ in an elementary way by showing the existence of a full Taylor expansion directly: we start with the Ansatz $u_1(x) = 1 + a_1 x + a_2 x^2 + \cdots$. We get $b(0)a_1 + (b'(0)/2 - 1) = 0$. Hence $a_1$. Then, inductively, we get an expression for $a_k$ as a function of the $a_l$ for $l < k$. Applying Malgrange's Theorem 7.1 in [10], we get a smooth solution $u_1$ with $u_1(0) = 1$.

Then we make the Ansatz $u_2 = u_1 v$ and we get the following differential equation for $v$:

$$\left( \mathrm{d}_x + \frac{a'}{a} + 2\frac{u_1'}{u_1} - i\frac{b}{a} \right) \mathrm{d}_x v = 0$$

It follows that, we can choose

$$\mathrm{d}_x v = \frac{1}{a u_1^2} e^{iE}$$

If $k$ is the order of the zero of $a$ at $x = 0$, we can choose local coordinates near 0 so that $E = 1/y^{k-1}$, we get $\mathrm{d}_y v = A(y) y^{-k} \exp(i/y^{k-1})$. We can integrate by part and we get

$$v(y) = A_0(y) e^{i/y^{k-1}} - \int_y^1 A_1(y) e^{i/y^{k-1}}$$

with $A_0$ and $A_1$ smooth. We can iterate the integration by part and get a formal solution $v(x) \equiv (v_0 + v_1 x + \cdots) e^{iE(x)}$. Again we can apply Malgrange's Theorem. □

## 5. Lorentzian Laplacians on surfaces

### 5.1. General facts on Lorentzian tori

We will consider for $X$ a 2-torus with a smooth Lorentzian metric $g$. Recall that a Lorentzian metric on a surface is a smooth non degenerate symmetric 2-form of signature $(1,1)$. There is, as in the Riemannian case, an associated geodesic flow (the Hamiltonian flow of the dual metric), a canonical volume form and a Laplace operator, which is an hyperbolic operator.

*The null curves.* — A smooth curve $\gamma$ is said to be *null* if, at every point $x$ of $\gamma$, and for any tangent vector $V$ to $\gamma$ at the point $x$, we have $g_x(V, V) = 0$, i.e. the tangent space to $\gamma$ at $x$ is isotropic for $g_x$. Locally the null curves are the leaves of two transverse foliations. This is not always true globally.

A *closed null leaf* $\gamma$ is a simple closed curve which is null. There is then a neighborhood of $\gamma$ with two null foliations: $\mathcal{F}_+$ is close to the tangent space to $\gamma$ and $\mathcal{F}_-$ is transversal to $\gamma$. We can then define a *Poincaré map* $P_\gamma$ as follows. Take a point $x_0$ on $\gamma$ and a germ of leaf of $\mathcal{F}_-$, $C$ at $x_0$. Then $P_\gamma$ is a germ of diffeomorphism $(C, x_0)$ into itself obtained by following the null leaves of $\mathcal{F}_+$. The map $P_\gamma$ is uniquely defined modulo smooth conjugation by germs of diffeomorphisms. Note that $P_\gamma$ is orientation preserving because $X$ is orientable.

### 5.2. Examples of non geodesically complete Lorentzian surfaces

It is known that Lorentzian metrics on the 2-torus are not always geodesically complete. It is the case for example for the Clifton-Pohl torus:

Let $T$ be the quotient of $\mathbb{R}^2 \setminus 0$ by the group generated by the homothety of ratio 2. On $T$, the Clifton-Pohl Lorentzian metric is $g := \mathrm{d}x\mathrm{d}y/(x^2+y^2)$. The associated Laplacian $\Box_g = (x^2+y^2)\partial^2/\partial x \partial y$ is formally self-adjoint on $L^2(T, |\mathrm{d}x\mathrm{dy}|/(x^2+y^2))$.

There is also a much simpler example, namely the quotient on $(\mathbb{R}^+_x \times \mathbb{R}_y, \mathrm{d}x\mathrm{d}y)$ by the group generated by $(x,y) \to (2x, y/2)$. The manifold is not closed, but non completeness sits already in a compact region.

It is known these metric are not geodesically complete. What about ESA of $\Box_g$?

### 5.3. Some results

We will prove a rather general result:

Theorem 5.1. —

(1) *If the metric $g$ admits a closed null leaf for which the Poincaré section is not tangent to infinite order to the identity, then $g$ is not geodesically complete. Under the same assumptions, $\Box_g$ is not ESA.*
(2) *If $g$ is conformal to a flat metric with a smooth conformal factor on a 2-torus, then $\Box_g$ is ESA.*

*Remark 5.2.* — In the first case, the proof of the null-incompleteness of the geodesic flow is due to Yves Carrière and Luc Rozoy [2]. We will reprove it.

Note that the conformal class of a Lorentzian metric is determined by the null foliations; hence ESA is a property of these foliations.

For the proof of Theorem 5.1, we will need two lemmas:

Lemma 5.3. — *The null-geodesic completeness is invariant by conformal change.*

*Proof.* — If $g = e^{\phi} g_0$, the dual metric satisfy $g^\star = e^{-\phi} g_0^\star$ and hence the geodesic flow restricted to $g^\star = 0$ are conformal with a bounded ratio. □

Lemma 5.4. — *The ESA property is invariant by conformal change.*

*Proof.* — If $\Box_g u = v$, we have also $\Box_{g_0} u = e^{\phi} v$ and $e^{\phi} v$ is in $L^2$ as soon as $v$ is. Hence, if $\Box_{g_0}$ is ESA, there exists a sequence $(u_n, w_n = \Box_{g_0} u_n)_{n\in\mathbb{N}}$ converging in $L^2$ to $(u, e^{\phi} v)$ and $e^{-\phi} v_n$ converges to $v$. □

This proves part (2) of Theorem 5.1.

## 5.4. Normal forms

It is well known and due to Sternberg [13] that a smooth germ of map $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$ whose differential at the origin is in $]0, 1[\cup]1, +\infty[$ is smoothly conjugated to $y \to \lambda y$ and hence is the time 1 flow of the vector field $\mu y \partial_y$ with $\lambda = e^{\mu}$.

A similar result hold for more degenerate diffeomorphisms: we assume that $g$ admits a closed null-leaf so that the Poincaré map is of the form $P(y) = y + y^k R(y)$ where $R(0) \neq 0$ and $k \geqslant 2$. It is proved in [15, Theorem 4].

Theorem 5.5. — *Any such map is the flow at time 1 of a vector field $V = A(y)\partial_y$ with $A \sim A_0 y^k$.*

Let $\gamma$ be closed null-leaf of $g$ and $U$ a neighbourhood of $\gamma$ so that we have the two null foliations $\mathcal{F}_+$ and $\mathcal{F}_-$. We have the:

Theorem 5.6. — *Let $\gamma$ be a closed leaf whose Poincaré map is $P = \mathrm{Id} + R$ with $R$ of order $k$. There exists coordinates near $\gamma$ so that the metric $g$ is conformal to $g_0 = \mathrm{d}x(\mathrm{d}y - a(y)\mathrm{d}x)$ with $a(y) \sim a_0 y^k, a_0 \neq 0$.*

*Proof.* — Let us parametrize the closed leaf $\gamma$ by $x \in \mathbb{R}/\mathbb{Z}$ and extend the coordinate $x$ in some neighbourhood $U$ of $\gamma$ so that the null foliation $\mathcal{F}_-$ is given by $\mathrm{d}x = 0$. Choose then for $y$ any coordinate in $U$ so that $y = 0$ on $\gamma$. We introduce the differential equation $\mathrm{d}y/\mathrm{d}x = A(x, y)$ associated to the foliation $\mathcal{F}_+$ close to $\gamma$. Note that $A(x, 0) = 0$. Let $\phi_x(y)$ be the flow of this differential equation. The map $y \to \phi_1(y)$ is the Poincaré map of $\gamma$. By Theorem 5.5, we can choose a vector field $a(y)\partial y$ so that the time one flow is the same Poincaré map; and denote by $(\phi_0)_x(y)$ this flow. Let us consider the germ of diffeomorphism near $\gamma$ defined by

$$F : (x, y) \to \big(x, y' = (\phi_0)_x \circ \phi_x^{-1}(y)\big).$$

The map $F$ sends the integral curves of $\mathrm{d}y - b\mathrm{d}x$ onto the integral curves of $\mathrm{d}y - a\mathrm{d}x$ and is periodic of period 1 because the time 1 flows are the same. Hence the two null foliations are given respectively by $\mathrm{d}x = 0$ and $\mathrm{d}y' - a(y')\mathrm{d}x = 0$. The Theorem follows. □

## 5.5. Proof of Theorem 5.1 (1)

The idea is to use the normal form which, being invariant by translation in $x$, allows a separation of variables and hence application of Theorem 4.1.

Let us first prove the null incompleteness. Using the normal form and the conformal invariance of null completeness, we have to study near $y = 0$ the Hamiltonian $h = -\eta(a(y)\eta + \xi)$. The function $\xi$ is a constant of the motion. Let us take initial conditions with $y_0 > 0$, $\xi_0 > 0$, and $a(y_0)\eta_0 + \xi_0 = 0$. We have, using that $a(y)\eta + \xi_0$ stays at 0, $\mathrm{d}y/\mathrm{d}t = 2a(y)\eta = -2\xi_0$. Hence $y(t)$ vanishes for a finite time $t_0$ and, we have then $\eta(t_0) = \infty$. Null incompleteness follows.

The Lorentzian Laplacian associated to $g = \mathrm{d}x(\mathrm{d}y - a(y)\mathrm{d}y)$ is given by

$$\Box = \partial_y a(y)\partial_y + \partial^2_{xy}$$

Let us look at solutions of $\Box u = v$ of the form $u(x, y) = e^{2\pi i x} v(y)$ with $v$ compactly supported near 0. We have

$$\Box u(x, y) = e^{2\pi i x} \left(\partial_y a(y)\partial_y + 2i\pi\partial_y\right) v(y)$$

The operator $P := \partial_y a(y)\partial_y + 2i\pi\partial_y$ is a Sturm–Liouville operator already studied in Section 3. $P$ is not ESA. It follows then that there exists $v$ compactly supported near 0 and $L^2$ so that $Pv = w \in L^2$ and there is no sequences $(v_n, w_n = Pv_n)$ converging in $L^2 \times L^2$ to $(v, w)$. The result follows.

## 5.6. Genericity

The goal of this section is to show that, for a generic Lorentzian metric on the 2-torus, there exists at least one null closed curve $\gamma$ whose Poincaré map is hyperbolic, i.e. such that the differential of $P_\gamma$ at the point $x_0$ of $\gamma$ is not tangent to the identity. It follows that, for a generic metric on the 2-torus, the geodesic flow is not complete and the Lorentzian Laplacian is not ESA.

A $C^\infty-$generic property is a property which holds for an open dense subset of the metrics in the $C^\infty$ topology.

We have the

Proposition 5.7. — *The existence of a closed null hyperbolic curve is a $C^\infty-$generic property of Lorentzian metrics on 2-tori.*

The following argument is due to Etienne Ghys.

*Proof.* — The openness of the set of metric with a closed null hyperbolic geodesic is evident.

We say that *the metric $g$ splits* if the null leaves belong to two distincts foliations $\mathcal{F}_+$ and $\mathcal{F}_-$. We say that *the metric $g$ is orientable* if there is a smooth non vanishing vector field $V$ on $X$ so that $g(V,V)$ is strictly positive everywhere. Any orientable metric splits. The two foliations are the boundaries of the connected component $C_+$ of the cone $g>0$ containing $V$. Indeed using an orientation of $X$, we choose $\mathcal{F}_+$ so that the frame generated by $\mathcal{F}_+$ and $V$ is positively oriented. We now study the two different cases.

*Case 1; $g$ splits.* — the genericity then follows from the fact that having an hyperbolic closed leaf is a generic property for a foliation of a torus (see [11]), here for $\mathcal{F}_+$.

*Case 2: $g$ do not split.* — We introduce in this case a two-fold cover $Y$ of $X$ for which the lift $G$ of the metric $g$ is orientable. This cover $Y$ is equipped with an involution $J$ exchanging the two null foliations. Let us take a null closed curve $\gamma$ of one of these foliations. Then $J(\gamma)$ is a null closed curve of the other. They cannot cross: they have the same rotation number, because $J$ is homotopic to the identity. Moreover all intersections have the same sign because both foliations as well as $Y$ are orientable. It follows that the projection of $\gamma$ onto $X$ is simple. Still having a closed hyperbolic leaf for the foliation $\mathcal{F}_+$ of $G$ is generic and $\mathcal{F}_- = J(\mathcal{F}_+)$. □

## 6. Further questions

There are still several open problems in this setting; we see at least five of them:

(1) Prove our Conjecture 1.2.
(2) Describe the self-adjoint extensions in the case of Lorentzian tori in a geometrical way.
(3) If we choose a self-adjoint extension, are there interesting spectral asymptotics?
(4) Extend to higher dimensional Lorentzian manifolds
(5) Extend to pseudo-differential operators of principal type.

## Appendix A. Regular singular points of linear differential equations of order two

For this section, one can look at [3] and [16].

We consider a linear differential equqation

$$Pu := (a(x)\mathrm{d}_x^2 + b(x)\mathrm{d}_x + c(x))u = 0$$

We assume that $a(0) = 0$ and 0 is a zero of finite order $k$ of $a$. The singular point $x = 0$ of $P$ is said to be *regular* if $b$ (resp. $c$) vanishes at order at least $k-1$ (resp. $k-2$) at $x = 0$. Otherwise 0 is an *irregular* singular point.

If $x = 0$ is a regular singular point, we introduce the *indicial equation*:

$$a^{(k)}(0)r(r-1) + kb^{(k-1)}(0)r + k(k-1)c^{(k-2)},\ r \in \mathbb{C}$$

We call $r_1,\ r_2$ the two roots of the indicial equation. Then the following holds:

- If $\operatorname{Im}(r_1 - r_2) \notin \mathbb{Z}$, there exist two independent solutions of $Pu = 0$ on a small interval $]0, c[$ of the form $u_j = x_+^{r_j} v_j(x)$
- If $\operatorname{Im}(r_2 - r_1) \in \mathbb{N}$, we have $u_1 = x_+^{r_1} v_1(x)$ and $u_2 = x_+^{r_2}(v_2(x)\log x + v_3(x))$

where the functions $v_j$ are smooth on $[0, c[$ and $v_1(0) = v_2(0) = 1$.

## Bibliography

[1] M. Braverman, O. Milatovic & M. Shubin, "Essential self-adjointness of Schrödinger-type operators on manifolds", *Usp. Mat. Nauk* **57** (2002), no. 4, p. 3-58, translation in *Russian Math. Surveys* **57** (2022), no. 4, p. 641–692.

[2] Y. Carrière & L. Rozoy, "Complétude des métriques Lorentziennes de $\mathbb{T}^2$ et difféomorphismes du cercle", *Bol. Soc. Bras. Mat.* **25** (1994), no. 2, p. 223-235.

[3] E. A. Coddington & N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955.

[4] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 82, Cambridge University Press, 1989.

[5] K. O. Friedrichs, "The Identity of Weak and Strong Extensions of Differential Operators", *Trans. Am. Math. Soc.* **55** (1944), p. 132-151.

[6] M. P. Gaffney, "A special Stoke's theorem for complete Riemannian manifolds", *Ann. Math.* **60** (1954), p. 140-145.

[7] L. Gårding, T. Kotake & J. Leray, "Uniformisation et développement asymptotique de la solution du problème de Cauchy linéaire, à données holomorphes; analogie avec la théorie des ondes asymptotiques et approchées (Problème de Cauchy I bis et VI)", *Bull. Soc. Math. Fr.* **92** (1964), p. 263-361.

[8] L. Hörmander, "$L^2$-estimates and existence theorems for the $\bar{\partial}$ operator", *Acta Math.* **113** (1965), p. 89-152.

[9] N. Lerner, "Pseudo-differential operators of degree one are ESA", manuscript, 2019.

[10] B. Malgrange, "Sur les points singuliers des équations différentielles", *Enseign. Math.* **20** (1974), p. 147-176.

[11] J. Palis & W. de Melo, *Geometric Theory of Dynamical Systems. An Introduction*, Springer, 1982.

[12] M. Reed & B. Simon, *Methods of modern mathematical physics. II: Fourier analysis, self- adjointness*, Academic Press Inc., 1975.

[13] S. Sternberg, "Local $C^n$ transformations of the real line", *Duke Math. J.* **24** (1957), p. 97-102.

[14] K. Taira, "Equivalence of classical and quantum completeness for real principal type operators on the circle", https://arxiv.org/abs/2004.07547, 2020.

[15] F. Takens, "Normal forms for certain singularities of vectorfields", *Ann. Inst. Fourier* **23** (1973), no. 2, p. 163-195.

[16] W. Wasow, *Asymptotic expansions for ordinary differentiel equations*, Pure and Applied Mathematics, vol. 14, Interscience Publishers, 1965.