# AROUND THE QUANTUM LENARD-BALESCU EQUATION

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Abstract. In the mean-field regime, a gas of quantum particles with Boltzmann statistics can be described by the Hartree-Fock equation. This dynamics becomes trivial if the initial distribution of particle is invariant by translation. However, the first correction is given on time of order $O(N)$ by the quantum Lenard–Balescu equation. In the first part of the present article, we justify this equation until time of order $O((\log N)^{1-\delta})$ (for any $\delta \in (0,1)$).

A similar phenomenon exists in the classical setting (with a similar validity time obtained by Duerinckx [Due21]). In a second time, we prove the convergence for dimension $d \geq 2$ of the solutions of the quantum Lenard–Balescu equation to the solutions of its classical counterpart in the semi-classical limit. This problem can be interpreted as a grazing collision limit: the quantum Lenard–Balescu equation looks like a cut-off Boltzmann equation, when the classical one looks like the Landau equation.

## Contents



## 1. Introduction

1.1. **General presentation.** Fix $L > 0$. We denote $\mathbb{T}_L$ the torus of length $L$, $(\mathbb{R}/L\mathbb{Z})^d$ (with $d \geq 2$) and $\mathfrak{H}_L$ the Hilbert space $L^2(\mathbb{T}_L)$. We want to describe $N$ particles in $\mathbb{T}_L$ interacting *via* pairwise potential energy. In the following, we denote $\mathbb{Z}_L^d := (2\pi L\mathbb{Z})^d$ the dual space of $\mathbb{T}_L$.

We fix $L$ and $N$ such that $NL^{-d} = 1$ (the density is constant). The state $\psi_N(t) \in \mathfrak{H}_L^N$ follows the Schrodinger dynamics

$$i\hbar\partial_t\psi_N(t) = \frac{\hbar^2}{2}\sum_{j=1}^N -\Delta_j\psi_N(t) + \frac{1}{N}\sum_{j<k}^N V_{j,k}\psi_N(t) \tag{1.1}$$

with $\Delta_j$ the Laplacian with respect to the $j$-th variable and $V_{j,k}$ the multiplication by $\mathcal{V}(x_j - x_k)$. For the moment, one can suppose $\mathcal{V}$ smooth, with spherical symmetry and compact support.

With a statistic point of view, the system is described by a density matrix $F_N \in \mathcal{L}^1(\mathfrak{H}_L)$ (the space of trace class operators, see Section 1.9 of [Gol13] for the definition and basic properties). An easy way to introduce $F_N$ is to understand it as a Hilbert-Schmidt operator with Kernel $F_N(x_{[1,N]}, y_{[1,N]}) \in L^2(\mathfrak{H}_L^{2N})$, with $x_{[1,N]} := (x_1, \cdots, x_N) \in \mathbb{T}_L^N$

$$F_N(x_{[1,N]}, y_{[1,N]}) := \int \psi(x_{[1,N]})\bar{\psi}_N(y_{[1,N]})\pi\,(\mathrm{d}\psi) \tag{1.2}$$

where $\pi$ is a probability measure on $\{\psi \in L^2(\mathfrak{H}_L^N), \|\psi\| = 1\}$.

As particles are exchangeable, the matrix $F_N$ has to verify the following (weak) symmetry condition:

$$\forall \sigma \in \mathfrak{S}_N,\ F_N(x_{\sigma(1)}, \cdots, x_{\sigma(N)}, y_{\sigma(1)}, \cdots, y_{\sigma(N)}) = F_N(x_1, \cdots, x_N, y_1, \cdots, y_N). \tag{1.3}$$

**Remark 1.1.** *We say that $F_N$ is*

- *Fermionic if $\pi$ has support included in (with $\varepsilon(\sigma)$ the signature of $\sigma \in \mathfrak{S}_n$)*
$$\{\psi \in \mathfrak{H}_L^N,\ \psi(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) = \epsilon(\sigma)\psi(x_1, \cdots, x_N)\},$$
$$\text{id est } F_N(x_1, \cdots, x_N, y_{\sigma(1)}, \cdots, y_{\sigma(N)}) = \epsilon(\sigma)F_N(x_1, \cdots, x_N, y_1, \cdots, y_N),$$
- *Bosonic if $\pi$ has support included in*
$$\{\psi \in \mathfrak{H}_L^N,\ \psi(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) = \psi(x_1, \cdots, x_N)\},$$
$$\text{id est } F_N(x_1, \cdots, x_N, y_{\sigma(1)}, \cdots, y_{\sigma(N)}) = F_N(x_1, \cdots, x_N, y_1, \cdots, y_N).$$

Using (1.1) and (1.2), the density matrix $F_{\hbar,N}(t)$ is solution of the Von Neumann equation

$$i\hbar\partial_t F_{\hbar,N}(t) = \frac{\hbar^2}{2}\sum_{j=1}^N [-\Delta_j, F_{\hbar,N}(t)] + \frac{1}{N}\sum_{j<k}^N [V_{j,k}, F_{\hbar,N}(t)],\ F_{\hbar,N}(t=0) := F_{N,0}. \tag{1.4}$$

At time 0, we make a *chaos assumption*, *i.e.* we suppose the factorization of the density matrix: there exists $F_0 \in \mathcal{L}^1(\mathfrak{H}_L)$ such that the density matrix $F_{N,0}$ verifies

$$F_{N,0}(x_1, \cdots, x_n, y_1, \cdots, y_n) = \prod_{i=1}^N F_0(x_i, y_i). \tag{1.5}$$

**Remark 1.2.** *Let $F_N$ be a Bosonic density matrix.*

- *if there exists $F_1 \in \mathcal{L}^1(\mathfrak{H}_L)$ such that $F_N = F_1^{\otimes N}$, then there exists $\psi \in \mathfrak{H}$ such that $F_1(x,y) = \psi(x)\bar{\psi}(y)$,*
- *if $F_1$ is translation invariant, then there exists a family $(c_k)_k$ such that*
$$F_1(x,y) = \sum_{k\in\mathbb{Z}_L} c_k e^{ik(x-y)}.$$

*We deduce that if $F_N$ is symmetric and invariant by translation, then there exists $k \in \mathbb{Z}_L^d$ such that*

$$F_{\hbar,N}(x_{[1,N]}, y_{[1,N]}) = \frac{1}{L^d}\exp\left(ik \cdot \sum_{j=1}^N (x_j - y_j)\right),$$

*which is a fix point of the mean-field dynamic. This shows that our results cannot be applied to Bosons.*

• **Derivation of an effective equation on the long time scale.** In that setting, one can prove (see [Gol13, PPS19]) that when $N \to \infty$, the density matrix of a typical particle $F_{\hbar,N}^{(1)}(t, x_1, y_1)$, where

$$\forall n \in [1,N],\ F_{\hbar,N}^{(n)}(t, x_{[1,n]}, y_{[1,N]}) := \int F_{\hbar,N}(t, x_{[1,n]}, z_{[n+1,N]}, y_{[1,n]}, z_{[n+1,N]})\,\mathrm{d}z_{[n+1,N]} \tag{1.6}$$

converges to $F_\hbar(t,x,y)$, the solution of the solution of Hartree equation

$$\begin{cases} i\hbar\partial_t F_\hbar(t) = \frac{\hbar^2}{2}[-\Delta, F_\hbar] + \left[\tilde{\mathcal{V}}, F_\hbar\right],\ F_\hbar(t=0) = F_0 \\ \tilde{\mathcal{V}}(t,x) = \int \mathcal{V}(x-y)F_\hbar(t,y,y)\,\mathrm{d}y. \end{cases}, \tag{1.7}$$

where $\tilde{\mathcal{V}}$ is an effective potential.

In [PPS19], the authors prove a stronger result, the propagation of chaos: if at time 0 the system is distributed with respect to $F_{0,N} := F_0^{\otimes N}$, then $\forall t \geq 0, j \in [1,N]$,

$$\sup_{y_{[1,j]}\in\mathbb{T}_L^j} \int \left|\left(F_{\hbar,N}^{(j)} - F_\hbar^{\otimes j}\right)(t, x_{[1,j]}, x_{[1,j]} + y_{[1,j]})\right| \mathrm{d}x_{[1,j]} \leq Ce^{C/\hbar t}\frac{j^2}{N} \tag{1.8}$$

where the constant $C$ does not depend on $\hbar$. Note that the norm is the $\mathcal{L}^1(\mathfrak{H}_L^j)$-operator norm.

If we choose an initial data invariant by translation ($F_0(x,y) = G(x-y)$), the effective potential becomes constant. Hence, at the first order, the density is constant at the first order.

However, the error terms $F^{(1)}_{\hbar,N}(t) - F_\hbar(t)$, $F^{(2)}_{\hbar,N}(t) - F_\hbar^{\otimes 2}(t)$ are of order $O(1/N)$. Hence it is natural to guess if one can obtain a corrector on long time (of order $O(N)$). Denoting $\Phi_{\hbar,N,L}(t,v)$ the Wigner transform of $F^{(1)}_{\hbar,N}$, scaled in time

$$\forall v \in \mathbb{Z}^d_{L/\hbar},\ \Phi_{\hbar,N,L}(t,v) := \frac{L^d}{(2\pi\hbar)^d}\int_{\mathbb{T}_{2L}} F^{(1)}_{\hbar,N}(Nt, \tfrac{y}{2}, -\tfrac{y}{2}) e^{i\frac{v}{\hbar}y}\,\mathrm{d}y \tag{1.9}$$

In limit $N\to\infty$, $L^dN = 1$, one can prove the formal convergence of $\Phi_{\hbar,N,L}(t,v)$ to the solution of the Quantum-Lenard-Balescu equation: for $c_d := \frac{2}{(2\pi)^d}$

$$\forall t \geq 0,\ \forall v_1 \in \mathbb{R}^d,\ \partial_t \Phi_\hbar(t,v_1) = Q^\hbar_{LB}(\Phi_\hbar(t))(v_1). \tag{1.10}$$

$$Q^\hbar_{LB}(\Phi)(v_1) = \frac{c_d}{\hbar^2}\int \frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(\Phi,k,v_1)|^2}\big(\Phi(v_2-\hbar k)\Phi(v_1+\hbar k) - \Phi(v_1)\Phi(v_2)\big)\,\mathrm{d}k\,\mathrm{d}v_2 \tag{1.11}$$

$$\varepsilon_\hbar(\Phi,k,v) = 1 + \hat{\mathcal{V}}(k)\int \frac{\Phi(v_*) - \Phi(v_* - \hbar k)}{\hbar k\cdot(v_* - v - \hbar k) - i0}\,\mathrm{d}v_*. \tag{1.12}$$

We precise that we take the convention for the Fourier transform

$$\hat{\mathcal{V}}(k) = \int \mathcal{V}(x) e^{-ikx}\,\mathrm{d}x. \tag{1.13}$$

This equation looks like the Boltzmann equation, with a dynamical screening $|\varepsilon_\hbar(\Phi,k,v_1)|^{-2}$, and verifies formally the same conservation laws and H-theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi_\hbar(t,v_1)\begin{pmatrix}1\\ v_1\\ |v_1|^2\end{pmatrix}\mathrm{d}v_1 = 0,\ \frac{\mathrm{d}}{\mathrm{d}t}\int \Phi_\hbar(t,v_1)\log\Phi_\hbar(t,v_1)\,\mathrm{d}v_1 \leq 0.$$

Hence, it describes an quantitative irreversibility coming from the mixing of the system.

This equation has been first formally derived by Balescu [Bal61] for particles in the Bose-Einstein or Fermi-Dirac statistic (for these particles, he found a cubic version of the equation).

Note that in the weak coupling scaling, a quantum system can also be described by a *collisional kinetic* equation. In that setting, the interaction are given by the short range potential: the density matrix respect the equation

$$i\hbar\partial_t F_N = -\hbar^2\sum_{i=1}^N [\Delta_i, F_N] + \sqrt{\hbar}\sum_{i<j}^N\left[\mathcal{V}\left(\tfrac{x_i-x_j}{\hbar}\right), F_N\right]. \tag{1.14}$$

with the scaling $N\hbar^3 = 1$ (in dimension 3). The equation has been first formally derived by Nordheim [Nor28], and Uehling and Uhlenbeck [UU33] (see also [Pu06]).

• **The classical counterpart.** A similar phenomenon exists in the classical setting. Consider a gas of $N$ particles inside the domain $\mathbb{T}_L$, with constant density $NL^d = 1$. Denoting their coordinate $(x_1,\cdots,x_N,v_1,\cdots,v_N)$ following the classical dynamic linked to the energy

$$\mathcal{H}_N := \sum_{i=1}^N \frac{|v_i|^2}{2} + \frac{1}{N}\sum_{i<j}^N \mathcal{V}(x_i - x_j). \tag{1.15}$$

We denote $F_N(t, x_{[1,n]}, v_{[1,n]})$ the distribution of particle. It is supposed initially factorized:

$$\tilde{F}_N(t=0, x_{[1,n]}, v_{[1,n]}) := \tilde{F}_0(x_1,v_1)\cdots\tilde{F}_0(x_N,v_N)$$

for some probability density $F_0$ on $\mathbb{T}_L\times\mathbb{R}^d$. Then one can show that the first marginal

$$\tilde{F}^{(1)}_N(t,x_1,v_1) := \int \tilde{F}_N(t,x_{[1,n]},v_{[1,n]})\,\mathrm{d}x_{[2,n]}\,\mathrm{d}v_{[2,n]}$$

converges when $N$ goes infinity to $\tilde{F}(t,x,v)$) the solution of the Vlasov equation (see Braun et Hepp [BH77], Dobrushin [Dob79], and a review of Golse [Gol22])

$$\begin{cases} \partial_t \tilde{F}(t) = -v\cdot\nabla_x \tilde{F}(t) + \nabla_x \tilde{\mathcal{V}}\cdot\nabla_v \tilde{F}(t), \ \tilde{F}(t=0) = \tilde{F}_0 \\ \tilde{\mathcal{V}}(t,x) = \int \mathcal{V}(x-y)\tilde{F}(t,y,v)\,\mathrm{d}y\,\mathrm{d}v. \end{cases}, \tag{1.16}$$

If we suppose that the initial density is inariant by translation, $F_0(x,v) := L^{-d}\Phi_0(v)$, then effective potential $\tilde{\mathcal{V}}$ become constant, and the Vlasov dynamics is trivial. As in the quantum case, one show that the scaled density

$$\Phi_N(t,v_1) := \int F_N(Nt, x_{[1,n]}, v_{[1,n]})\,\mathrm{d}x_{[1,n]}\,\mathrm{d}v_{[2,n]}$$

converges as $N\to\infty$, $NL^d = 1$ to $\Phi(t,v_1)$, solution of the (classical) Lenard-Balescu equation

$$\partial_t \Phi = Q^0_{LB}(\Phi)(v_1) \tag{1.17}$$

$$Q^0_{LB}(\Phi)(v_1) := \nabla_{v_1}\cdot\int B(\Phi,v_1,v_2)\left(\Phi(v_2)\nabla\Phi(v_1) - \Phi(v_1)\nabla\Phi(v_2)\right)\mathrm{d}v_2 \tag{1.18}$$

$$B(\Phi,v_1,v_2) := c_d\int \frac{|\hat{\mathcal{V}}(k)|^2 k\otimes k}{|\varepsilon_0(\Phi,k,v)|^2}\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k \tag{1.19}$$

$$\varepsilon_0(\Phi,k,v) := 1 + \hat{\mathcal{V}}(k)\int \frac{k\cdot\nabla\Phi(v_*)}{k\cdot(v-v_*) - i0}\,\mathrm{d}v_*. \tag{1.20}$$

The equation has been derived simultaneously in the 60s by Guernsey [Gue61, Gue62], Lenard [Len60] and [Bal60]. In the former two derivation, the authors use a truncation of the BBGKY hierarchy. This strategy has been by adapted by Duerinckx [Due21] and Duerinckx–Saint-Raymond [DSR21] in order to obtain a consistency result. Note that in the first result, the author reach the time $O((\log N)^{1-\delta})$ (compare to $O(N)$ of the kinetic time), when in [DSR21], the authors reaches the longer time $O(N^\delta)$ in a linear setting (where $\delta\in(0,1)$ is small enough).

The classical Lenard-Balescu equation has been studied by Duerinckx and Winter in [DW23].

• **Semi-classical limit of the quantum Lenard–Balescu equation.** In the semi-classical limit $\hbar\to 0$, one can proof that the quantum system "converge" to the classical one. More precisely, consider $F_\hbar(t,x,y)$ solution of the Hartree equation with initial data $F_{0,\hbar}(x,y)$. At time 0, we suppose that the Wigner transform of $F_{0,\hbar}$ defined by

$$\tilde{F}_{\hbar,0}(x,v) := \frac{1}{(2\pi\hbar)^d}\int F_{0,\hbar}(x+\tfrac{y}{2}, x-\tfrac{y}{2})e^{i\frac{v}{\hbar}y}\,\mathrm{d}y \tag{1.21}$$

converges to some distribution $\tilde{F}_0(x,v)$. Then the Wigner transform of $F_\hbar(t,x,y)$ converge to $\tilde{F}(t,x,v)$, the solution of the Vlasov equation with initial data $F_0(x,v)$.

One can ask a similar question for the quantum Lenard–Balescu equation: for $\Phi_\hbar(t,v_1)$ the solutions of the quantum Lenard–Balescu (1.10) with initial data $\Phi_0(v_1)$ equation converge to $\Phi(t,v_1)$, the solution of the classical Lenard–Balescu equation (1.17) with the same initial data.

This problem looks like a grazing collision limit: we want to link a Boltzmann like operator

$$Q^\hbar(\Phi)(v_1) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}\left(\Phi\left(\tfrac{v_1+v_2}{2} + \tfrac{|v_1-v_2|}{2}\sigma\right)\Phi\left(\tfrac{v_1+v_2}{2} - \tfrac{|v_1-v_2|}{2}\sigma\right) - \Phi(v_1)\Phi(v_2)\right)B_\hbar(\Phi,v_1,v_2,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_2 \tag{1.22}$$

toward a Landau type operator

$$Q^0(\Phi)(v_1) = \nabla_1\cdot\int_{\mathbb{R}^d}\left(\nabla\Phi(v_1)\Phi(v_2) - \Phi(v_1)\nabla\Phi(v_2)\right)B_0(\Phi,v_1,v_2)\,\mathrm{d}v_2. \tag{1.23}$$

if the collision kernel $B_\hbar(\Phi,\frac{v_1+v_2}{2},\frac{v_1-v_2}{2},\sigma)$ concentrates on the $\sigma$ with $\sigma\cdot\frac{v_1-v_2}{|v_1-v_2|}\simeq 1$. Here, we use the variable $\sigma := \frac{2\hbar k+v_1-v_2}{|v_1-v_2|}$, which is common in the literature.

The rigorous analysis of the grazing collision limit goes back to the classical result of Arsenev and Buryak in [AB91], and a general setting as been treated by Alexandre and Villani in [AV04]. However, the solutions consider by these two authors are very weak (they are called the *renormalized solution*). As the Lenard-Balescu kernel $B_\hbar(\Phi_\hbar(t),v_1,v_2,\sigma)$ depends on the solution $\Phi_\hbar(t)$, one needs a strong notion of solution.

In [He14], He proved the strong convergence of the solution of the Boltzmann equation toward the solution of the Landau equation, in the case where the kernel $B_\hbar(\Phi_\hbar(t),v_1,v_2,\sigma)$ is independent of $\Phi_\hbar(t)$ and $\frac{v_1+v_2}{2}$, and that is not integrable with respect to the variable $\sigma$ (these kernels are said *uncutoff*). In that case, the Boltzmann operator acts like a fractional Laplacian, bringing some additional regularity.

Note also the work of He, Lu, Pulvirenti and Ma around the classical limit of the quantum Boltzmann equation [HLP21, HLPZ24].

In the case of Lenard–Balescu kernel, there is *cutoff*: for $\hat{\mathcal{V}} \in L^1 \cap L^\infty(\mathbb{R}^3)$, one has

$$\int \hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k = \frac{\mathfrak{c}}{\hbar^2|v_1-v_2|}\int_0^{\frac{|v_1-v_2|^2}{2\hbar^2}} \hat{\mathcal{V}}^2(y)\,\mathrm{d}y$$

for some constant $\mathfrak{c}$. In the present paper, we propose a strategy that can be applied to cutoff kernel, for any dimension $d \geq 2$.

## 1.2. Statement of the results.

1.2.1. *Short time validity of the quantum Lenard–Balescu equation.* We begin by justifying the quantum Lenard–Balescu equation. The main goal would be reaching the kinetic time $O(N)$, but this is for the moment out of reach. However, we are able to propose a partial result, up to some time $O((\log N)^{(1-\delta)})$ for any $\delta \in (0,1)$.

We consider an interaction potential $\mathcal{V}$ satisfying the following assumption:

**Assumption 1.** The interaction potential $\mathcal{V}$ is smooth, radial and with compact support.

**Theorem 1.** *Consider a density $\Phi_0$ such that*

$$\Phi_0 \geq 0, \quad \int \Phi_0(v)\,\mathrm{d}v = 1 \text{ and } \|\langle v\rangle^{2d}\langle\nabla\rangle^2\Phi_0\|_{L^\infty}\|\hat{\mathcal{V}}\|_\infty < \frac{1}{2}. \tag{1.24}$$

*At $t=0$ we set the system such that*

$$F_{0,L}(x,y) := \frac{(2\pi\hbar)^d}{L^{2d}\mathcal{Z}_L}\sum_{k\in\mathbb{Z}_L}\Phi_0(\hbar k)e^{ik\cdot(x-y)}, \tag{1.25}$$

$$\hat{F}_{N,0} = F_{0,L}^{\otimes N}. \tag{1.26}$$

*where $\mathcal{Z}_L$ is a normalization constant, and we construct $F_N(t) : \mathbb{R}^+ \to \mathcal{L}^1(\mathfrak{H}_L^N)$ the solution of the Von Neumann equation* (1.4) *with initial data $F_{N,0}$.*

*We fix the scaling $N\to\infty$, $L\to\infty$ with $L = N^\gamma$ for any $\gamma > 0$. Then for any diverging sequence $\tau_N \ll (\log N)^\delta$ (for some $\delta \in (0,1)$ to be fixed) and any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}_t^+ \times \mathbb{R}_v^d)$, the scaled Wigner transform $\Phi_{N,L}$ verifies*

$$\int_0^\infty \frac{(2\pi\hbar)^d}{L^d}\sum_{v\in\mathbb{Z}_{L/\hbar}}\psi(t,v)\,\partial_t\Phi_{N,L}(\tfrac{\tau_N}{N}t,v)\,\mathrm{d}t \underset{\substack{N\to\infty\\ L=N^\gamma}}{\longrightarrow} \int_0^\infty\int \psi(t,v)Q_{LB}^\hbar(\Phi_0)(v)\,\mathrm{d}v\,\mathrm{d}t. \tag{1.27}$$

The proof is given in Section 2.

We follow mostly the strategy of [Due21] in the classical setting. It is based on the estimation of the "defect of factorization" (also called the *cumulant*, introduce in (2.1)) of the marginal of the system. One can prove that for short time (of order $O((\log N)^{1-\delta})$), the correction implying three or more particles can be neglected. The remaining system of two equations gives the Lenard–Balescu evolution.

In [Due21], the estimations of the cumulants are based on the Glauber calculus. In the present paper, we use an improvement of the cumulant estimates of [PPS19].

Note that in general, obtaining a justification of a collisional kinetic equation from a particle system is a difficult problem. The only full (non-linear, until a kinetic time) results hold in the low density setting, for the hard spheres system [Lan75, Kin75, IP89, Spo91, GSRT13, PSS14, DHM24], and for the interacting wave system [DH21, DH23]. The other results are either in a *linear setting* (when we follow a particle in a bath at equilibrium [LLSvB80, DP99, DR01, BGSR16, Ayi17, Cat18, MS24]) or in the linearized setting (the study of the fluctuations of the empirical measure around the equilibrium [Spo81, Spo83, BGSR17, BGSRS21, BGSRS24, LB24]), or are consistency result (result at time 0) [BPS13, Win21, VW18, Due21, DR01].

1.2.2. *Property of quantum Lenard–Balescu equation.* We study now the property of quantum Lenard–Balescu equation. We note that the Maxwellian

$$M(v) := \tfrac{1}{(2\pi)^{d/2}}\exp(-\tfrac{|v|^2}{2})$$

is a steady state of the equation. We want to look at a solution close to this equilibrium:

$$F(t,v_1) = M(v_1) + \sqrt{M(v_1)}g(t,v_1). \tag{1.28}$$

The equation solved by $g(t, v_1)$ is

$$\partial_t g(t,v_1) + \mathcal{L}_\hbar g(t,v_1) = \mathcal{Q}_\hbar(g(t),g(t))(v_1) \tag{QLB$_\hbar$}$$

where we denote $F_h := M + \sqrt{M}h$ and

$$\mathcal{L}_\hbar g(v_1) := \frac{c_d}{\hbar^2}\int \Delta\!\!\!\Delta \left(g\sqrt{M_*}\right) \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\sqrt{M_2}\,\mathrm{d}k\,\mathrm{d}v_2 \tag{1.29}$$

$$\begin{aligned}\mathcal{Q}_\hbar(g,h)(v_1) := &-\frac{c_d}{\hbar^2}\int \Delta\!\!\!\Delta(gh_*)\frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(F_h,k,v_1)|^2}\sqrt{M_2}\,\mathrm{d}k\,\mathrm{d}v_2 \\ &+\frac{c_d}{\hbar^2}\int \Delta\!\!\!\Delta\left(g\sqrt{M_*}\right)\left(\frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2} - \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(F_h,k,v_1)|^2}\right)\sqrt{M_2}\,\mathrm{d}k\,\mathrm{d}v_2\end{aligned} \tag{1.30}$$

where $\mathcal{L}_\hbar$ is the Linearized Boltzmann operator. We use the following notation

$$\begin{gathered} v_1' := v_1 + \hbar k,\ v_2' := v_2 - \hbar k,\ \hat{k} := \tfrac{k}{|k|}, \\ \Delta\!\!\!\Delta gh_* := g(v_1)h(v_2) + g(v_2)h(v_1) - g(v_1')h(v_2') - g(v_2')h(v_1') \\ \forall i \in \{1,2\},\ j \in \{\ ,'\},\ f_i^j := f(v_i^j). \end{gathered}$$

It is well known that the operator $\mathcal{L}_\hbar$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$. We introduce the weighted Sobolev space $\mathcal{H}^r \subset L^2(\mathbb{R}^d)$, with norm

$$\|g\|_r^2 := \sum_{r'=0}^{r} \left\| \left(\nabla - \frac{v}{2}\right)^{r'} g \right\|^2 < \infty, \tag{1.31}$$

where we denote $\|\ \|$ the $L^2(\mathbb{R}^d)$. Note that the norm $\|g\|_r$ is equivalent to $\|(\nabla - \frac{v}{2})^r g\|$.

In the following, we choose a potential $\mathcal{V}$ satisfying the following assumption:

**Assumption 2.** We set the dimension $d \geq 2$. We define $\mathcal{V}$ a symmetric ($\mathcal{V}(x) = \mathcal{V}(|x|)$) such that there exists three constants $C > 0$ and $s, s' > d + 3$

$$\frac{1}{C} \leq |\hat{\mathcal{V}}(k)|^2(1+|k|)^s \leq C,\ |\hat{\mathcal{V}}'(k)|^2 + |\hat{\mathcal{V}}''(k)|^2 \leq \frac{C}{(1+|k|)^{s'}}. \tag{1.32}$$

**Notation** *As the constants in the inequalities change along the computation, we denote $A \lesssim B$ if there exists a constant $C > 0$ independent of $\hbar$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we denote $A \simeq B$. We denote $A = O(B)$ if $A \lesssim B$.*

• **Cauchy theory of the Quantum Lenard-Balescu Equation.**

The first step of the semi-classical limit is the construction of solutions with estimates independent of $\hbar$. One has a long time existence theorem for the (QLB$_\hbar$)-equation, but only for large dimension.

**Theorem 2.** *For any $d \geq 2$ and $r \geq 5$, there exists $\hbar_0 > 0$ and $\eta > 0$ such that for any initial data $g_0 \in \mathcal{H}^r$ with $\|g_0\|_r \leq \eta$ and any $\hbar < \hbar_0$, there exists a unique $g_\hbar \in \mathcal{C}_b^0(\mathbb{R}^+, \mathcal{H}^r)$ solution of the* (QLB$_\hbar$)*-equations.*

*In addition, one have the following bound:*

$$\sup_{t\geq 0} \|g_\hbar(t)\|_r \leq \|g_0\|_r. \tag{1.33}$$

The proof is given in Section 3. As we want to perform a semi-classical limit, we want to obtain a Cauchy theory closed to the one obtain by Duerinckx and Winter [DW23] (following the strategy of [Guo02]).

In classical case, Duerinckx and Winter construct a weighted Sobolev norm $\|\cdot\|_0$, controlling the non-linear operator. In order to use the dissipation coming from the linear operator $\mathcal{L}_0$, the norm $\|\cdot\|_0$ is bonded from below by $\mathcal{L}_0$: denoting $\pi_0$ the $L^2$-orthogonal projector onto the space $< \sqrt{M}, v_1\sqrt{M}, \cdots v_d\sqrt{M}, |v|^2\sqrt{M} >$,

$$\forall g \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \int g\mathcal{L}_0 g \leq C\|\pi_0 g\|_0^2$$

for some constant $C$.

In the quantum case, we construct a family of norm $\|\ \|_\hbar$, each controlling the non-linear operator $\mathcal{Q}_\hbar$. The $\|\ \|_\hbar$ converge to $\|\ \|_0$ (for $g$ smooth enough, $\|g\|_\hbar \to \|g\|_0$). We will prove that there exists a constant $C$ independent of $\hbar$ such that

$$\forall g \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \int g\mathcal{L}_\hbar g \leq C\|\pi_0 g\|_\hbar^2.$$

One can apply the same estimation than in the classical case.

In addition, one can obtain the following short time result:

**Theorem 3.** *For any dimension $d \geq 2$ and $\hbar_0 > 0$, $r \geq 5$, there exists to constant $c_0 > 0$ such that for any $\hbar < \hbar_0$*

*Fix $g_0 > 0$ an initial data, and a constant $C_{g_0}$ with the following bounds*

$$\varepsilon_\hbar(M + \sqrt{M} g_0, v, k) \geq \frac{1}{C_{g_0}}, \quad \|g_0\|_r \leq C_{g_0}. \tag{1.34}$$

*Then the* (QLB$_\hbar$) *equation admits a solution $g \in \mathcal{C}([0, \frac{c_0 \hbar^2}{C_{g_0}^{2r+4}}], \mathcal{H}^r)$.*

The proof is given in Section 5

• **Semi-classical limit.** One of the goals of the present paper is to describe the limit as $\hbar$ goes to zero. We want to link the quantum Lenard-Balescu with its classical counterpart

For perturbative solution $F(t) := M + \sqrt{M} g(t)$, the classical equation (1.17) becomes

$$\partial_t g(t, v_1) + \mathcal{L}_0 g(t, v_1) = \mathcal{Q}_0(g(t), g(t))(v_1) \tag{CLB}$$

where we denote $H := M + \sqrt{M} h$ and

$$\mathcal{L}_0 g(v_1) := \left(\nabla - \frac{v_1}{2}\right) \cdot \int B(M, v_1, v_2) \left[\sqrt{M_2}\left(\nabla_1 + \frac{v_1}{2}\right) g_1 - \sqrt{M_1}\left(\nabla_2 + \frac{v_2}{2}\right) g_2\right] \sqrt{M_2}\, \mathrm{d}v_2 \tag{1.35}$$

$$\begin{aligned} \mathcal{Q}_0(g, h)(v_1) := &- \left(\nabla - \frac{v_1}{2}\right) \cdot \int B(H, v_1, v_2) \left[h_2 \nabla g_1 + g_2 \nabla h_1 - h_1 \nabla g_2 - g_1 \nabla h_2\right] \sqrt{M_2}\, \mathrm{d}v_2 \\ &+ \left(\nabla - \frac{v_1}{2}\right) \cdot \int (B(H, v_1, v_2) - B(M, v_1, v_2)) \left[\sqrt{M_2}\left(\nabla_1 + \frac{v_1}{2}\right) g_1 - \sqrt{M_1}\left(\nabla_2 + \frac{v_2}{2}\right) g_2\right] \sqrt{M_2}\, \mathrm{d}v_2 \end{aligned} \tag{1.36}$$

**Theorem 4.** *For any $d \geq 2$ and $r \geq 5$, there exist $\hbar_0$ and $\eta > 0$ such that for any initial data $g_0 \in \mathcal{H}^r$ with $\|g_0\|_r \leq \eta$ and any $\hbar < \hbar_0$, the sequence of $(g_\hbar)_{\hbar < \hbar_0}$ of solution of the* (QLB$_\hbar$)*-equations converges weakly in $\mathcal{C}_b^0(\mathbb{R}^+, \mathcal{H}^r)$ as $\hbar \to 0$ to $g_\infty$, solution of the* (CLB)*-equation with initial data $g_\infty(t = 0) = g_0$.*

*In addition, one have the following bound:*

$$\forall t \geq 0, \ \|g_\hbar(t) - g_\infty(t)\| \lesssim \hbar t \|g_0\|_r \tag{1.37}$$

**Remark 1.3.** *The preceding estimates can be improve to get a bound on $\|g_\hbar(t) - g_\infty(t)\|_r$. As the computations become technical, we prefer to skip them.*

The proof is given in Section 4.

## 2. Validity at time 0 of the Quantum Lenard-Balescu equation

In the first part of the present paper, we present a system from which the quantum Lennard–Balescu equation can be derived (at least at time 0).

The strategy is quiet disconnected from the rest of the paper, and is mainly inspired from [PPS19] for the estimation of the cumulant, and from [Due21] for the computation of the collision operator.

We denote $\mathcal{L}^1(\mathfrak{H})$ the space of trace-class operator on the Hilbert space $\mathfrak{H}$, and $\mathfrak{H}_L := L^2(\mathbb{T}_L)$.

We recall that the system is describe by a density matrix $F_N(t) \in \mathcal{L}^1(\mathfrak{H}_L^N)$, self-adjoint, symmetric: for all $\sigma \in \mathfrak{S}_N$ and $(\psi_i)_{i \leq N} \in \mathfrak{H}_L^N$,

$$\langle \psi_1 \otimes \cdots \psi_N, F_N(t) \psi_1 \otimes \cdots \psi_N \rangle = \left\langle \psi_{\sigma(1)} \otimes \cdots \psi_{\sigma(N)}, F_N(t) \psi_{\sigma(1)} \otimes \cdots \psi_{\sigma(N)} \right\rangle.$$

It is solution of the Von-Neuman–Fock equation

$$i\hbar \partial_t F_N(t) = \frac{\hbar^2}{2} \sum_{j=1}^N [-\Delta_j, F_N(t)] + \frac{1}{N} \sum_{j<k}^N [V_{j,k}, F_N(t)].$$

Here, $\Delta_j$ is the Laplacian with respect to the $j$-th variable, and $V_{j,k}$ the multiplication by $\mathcal{V}(x_j - x_k)$.

At time 0, we fix

$$F_{0,L}(x, y) := \left(\frac{2\pi\hbar}{L^2}\right)^d \frac{1}{\mathcal{Z}_L} \sum_{k \in \mathbb{Z}_L} \Phi_0(\hbar k) e^{ik\cdot(x-y)},$$
$$\hat{F}^N(t = 0) = F_{0,L}^{\otimes N}.$$

2.1. *A priori* **estimation of the cumulant.** In the following we introduce the notations: for $f \in \mathcal{L}^1(\mathfrak{H}_L)$ and $\sigma \subset [1, N]$

$$f^\sigma := \bigotimes_{k\in\sigma} f_k \in \mathcal{L}^1(\mathfrak{H}_L^\sigma),$$

and $\mathrm{tr}_\sigma$ the trace with respect the particles in $\sigma \subset [n]$:

$$\forall A \in \mathcal{L}^1(\mathfrak{H}^N), B \in \mathcal{L}(\mathfrak{H}^\sigma),\ \mathrm{tr}\left(A\left(B \otimes \mathbb{1}^{[1,n]\setminus\sigma}\right)\right) = \mathrm{tr}\left(\mathrm{tr}_\sigma A\, B\right).$$

We use the notation $[m, n] = \{m, m+1, \cdots, n\}$ and $[n] := [1, n]$.
That defines the $k$-th marginal of the density matrix $F_N \in \mathcal{L}^1(\mathfrak{H}^N)$:

$$F_N^{[k]} := \mathrm{tr}_{[1,k]} F_N.$$

If the system start from a factorized initial data $F_{N,0} = F_0^{\otimes N}$, one can expect that the system remains almost factorized. As the interaction creates some correlation between particles it will never remains fully factorized and we introduce the family of density matrix $(G_{N,n})_n \in \prod_{n\geq 1} \mathcal{L}^1(\mathfrak{H}_L^n)$ to measure the defect of tensorisation:

$$G_N^{[n]}(t) = \sum_{\sigma\subset[n]} (-1)^\sigma G_N^{[n]\setminus\sigma}(t) \otimes (F_N^{(1)}(t))^\sigma. \tag{2.1}$$

Using the $g_\omega^N$, one can decompose the marginal as

$$F_N^{[n]}(t) = \sum_{\sigma\subset[1,n]} G_N^{[1,n]\setminus\sigma}(t) \otimes (F_N^{(1)}(t))^\sigma. \tag{2.2}$$

We will prove the following result, which is similar to Theorem 2.2 of [PPS19].

**Proposition 2.1.** *Let $(F_0^N) \in \prod_N \mathcal{L}^1(\mathfrak{H}_L^N)$ a family of initial data such that for all $N \geq 1$,*

$$\forall n \in [1, N], \|G_N^{[n]}(t=0)\|_{\mathcal{L}^1_s(\mathfrak{H}_L^n)} \leq A^n \left(\frac{n}{\sqrt{N}}\right)^{n/2} \tag{2.3}$$

*Then there exists a constant $C > 0$ independant of $C^0$ such that for any $\beta \in (0, 1/9)$,*

$$\forall \frac{t}{\hbar} \leq \frac{\beta}{C\|\mathcal{V}\|_\infty},\ \forall n \geq 1,\ \|G_N^{[n]}(t)| \leq (CA)^n \left(\frac{n^{1-\beta}}{N^{\frac{1-3\beta}{2}}}\right)^n.$$

*Proof.* We define $\tilde{F}(t)$ the solution of the Hartree equation

$$\begin{cases} i\hbar\partial_t \tilde{F}(t) = \hbar^2\left[-\Delta, \tilde{F}(t)\right] + \mathrm{tr}_1\left[V_{1,2}, \tilde{F}(t)\right] \\ \tilde{F}(t=0) = F_N^{(1)}(t=0) \end{cases}$$

We introduce the family of matrices $E_{[1,n]}^N(t)$ defined by

$$E_N^{[n]}(t) = \sum_{\sigma\subset[n]} (-1)^\sigma F_N^{[n]\setminus\sigma}(t) \otimes (\tilde{F}(t))^\sigma. \tag{2.4}$$

It has been proved in [PPS19] that for $t > 0$, there exists a constant $C > 0$ (independent of $N$ and $L$) such that for all $t > 0$,

$$\|F_N^{[1]}(t) - \tilde{F}(t)\| \leq \frac{C}{N} e^{C\frac{t}{\hbar}},\ \|E_N^{\{1,2\}}(t)\| \leq \frac{C}{N} e^{C\frac{t}{\hbar}}. \tag{2.5}$$

We deduce immediately that for $\beta \in (0, 1)$, $\forall \frac{t}{\hbar} \leq \frac{\beta}{C\|\mathcal{V}\|_\infty}$

$$\|G_N^{[2]}(t)\| \leq \|\tilde{F} \otimes \tilde{F} - F_N^{\{1\}} F_N^{\{2\}}\| + \|E_N^{[2]}\| \leq \frac{C}{N} e^{C\frac{\|\mathcal{V}\|_\infty t}{\hbar}} \leq \frac{C}{N^{1-\beta}}.$$

We can now look at the $G_N^{[n]}$. They are solution of the following hierarchy (its derivation is similar to the Appendix of [PPS19])

$$\begin{aligned} i\hbar\partial_t G_N^{[n]} = \left(\hbar^2 K_n + \frac{1}{N}\sum_{j<k} T_{j,k}\right) g_{[n]} + D_n(f_1) G_N^{[n]} + D_n^1(f_1) G_N^{[n+1]} \\ + \bar{D}_n^{-1}(F_N^{(1)}, G_N^{[2]}) G_N^{[n-1]} + D_n^{-2}(F_N^{[1]}) G_N^{[n-2]} \end{aligned} \tag{2.6}$$

where

$$K_n := \sum_{j=1}^{n} [-\tfrac{\Delta_j}{2}, \cdot],\ T_{j,k} := [V_{j,k}, \cdot],\ C_{j,n+1} h_{[n+1]} = \mathrm{tr}_{[1,n]} \Big( T_{j,n+1} h_{[n+1]} \Big)$$

$$\bar{D}_n^{-1}(f_1, g_2) h_{[n-1]} := \sum_{j \in [n]} \Big( C_{1,2} g_{[2]} \Big)^{\{j\}} g_{[n] \setminus \{j\}} + D_n^{-1}(f_1) h_{[n-1]}$$

$$D_N^1 = D_0^{-2} := 0$$

$$D_1^{-1}(f_1) h_\emptyset := -\frac{1}{N} C_{1,2} f_1 \otimes f_1$$

$$D_2^{-2}(f_1) h_\emptyset := \frac{1}{N} \Big( T_{1,2} f_1 \otimes f_1 - \big( C_{1,2}(f_1 \otimes f_1) \big) \otimes f_1 - f_1 \otimes \big( C_{1,2}(f_1 \otimes f_1) \big) \Big)$$

with the similar bound, for $\frac{t}{\hbar} \leq \frac{\beta}{C \|\mathcal{V}\|_\infty}$, $\beta \in (0, 1/2)$,

$$\|\bar{D}_N^{-1}\| \leq \|\mathcal{V}\|_\infty \Big( \frac{4n^2}{N} + n \|g_{[2]}\| \Big) \leq \|\mathcal{V}\|_\infty \Big( \frac{4n^2}{N} + \frac{n}{N} C e^{C \|\mathcal{V}\|_\infty \frac{t}{\hbar}} \Big) \leq C' \|\mathcal{V}\|_\infty \Big( \frac{n^2}{N} + \frac{n}{N^{1-\beta}} \Big)$$
$$\leq C' \|\mathcal{V}\|_\infty \Big( \frac{n^2}{N} + \Big( \frac{n^2}{N} \Big)^{1-\beta} \Big).$$

Note that this hierarchy of equation can be simplified considering th *Grand canonical ensemble*: the number of particles is not fixed but is a random (see [PPS22]).

In [PPS19], the authors use the following technical lemma, which is proved in Section 3 of [PPS19]

**Lemma 2.2.** *Let $(g_n(t))_{n \leq N}$, $g_n : \mathbb{R}^+ \to \mathcal{E}_n$ be a family of function onto some Banach spaces $\mathcal{E}_n$ and $\alpha \in (0,1)$ such that*

$$\forall t \geq 0,\ \|g_n(t)\| \leq 2^n,\ \forall n \geq 1,\ \|g_n(0)| \leq A^n \Big( \frac{n^2}{N} \Big)^{\frac{\alpha n}{2}}$$

$$\partial_t \|h_n\| \leq n \big( \|h_n\| + \|h_{n+1}\| \big) + \Big( \frac{n^2}{N} + \Big( \frac{n^2}{N} \Big)^\alpha \Big) \big( \|h_{n-1}\| + \|h_{n-2}\| \big)$$

*Then there exists a constant $C$ independent of $A$ such that*

$$\forall n \geq 1,\ \|g_n(t)| \leq \Big( C A e^{Ct} \Big)^n \Big( \frac{n^2}{N} \Big)^{\frac{\alpha n}{2}}.$$

Applying a second time the lemma, one obtain

$$\forall n \geq 1,\ \|G_N^{[n]}(t)| \leq \Big( C A e^{Ct} \Big)^n \Big( \frac{n^2}{N} \Big)^{\frac{(1-\beta)n}{2}} \leq (CA)^n \left( \frac{n^{1-\beta}}{N^{\frac{1-3\beta}{2}}} \right)^n.$$

□

## 2.2. Effective equation for initial data stable by translation.

In the classical setting, the Lenard-Balescu equation is a correction of the Vlasov dynamics in the case of uniformly distributed (in $x$) initial data. We have to precise the meaning in the quantum case.

*Definition* 2.1. We define for $z \in \mathbb{T}_L$ the translation operator $\tau_z : \mathfrak{H}_L^N \to \mathfrak{H}_L^N$ by

$$\forall \psi \in \mathfrak{H}_L^N,\ (\tau_z \psi)(x_1, \cdots, x_N) = \psi(x_1 + z, \cdots, x_N + z).$$

A density matrix $F^N \in \mathcal{L}^1(\mathfrak{H}_L^N)$ is said *invariant by translation* if it commutes with all the translation operators. We denote $\mathcal{L}_i^1(\mathfrak{H}_L^N) \subset \mathcal{L}^1(\mathfrak{H}_L^N)$ the set of operator invariant by translation.

We can write this property for density matrix in the $X$ variable: denoting $F^N(x_1, \cdots, x_N, y_1, \cdots, y_N)$ its kernel, $F_N$ is invariant by translation if

$$\forall z \in \mathbb{T}_L,\ F_N(x_1 + z, \cdots, x_N + z, y_1 + z, \cdots, y_N + z) = F_N(x_1, \cdots, x_N, y_1, \cdots, y_N).$$

This property can also be defined in the Fourier Variable: Denote $\mathbb{Z}_L := \big( \frac{2\pi}{L} \mathbb{Z} \big)^d \big)$, and for $(k_{[N]}, l_{[N]}) \in \mathbb{Z}_L^{2n}$

$$\hat{F}_N(k_{[N]}, l_{[N]}) := \iint F_N(x_{[N]}, y_{[N]}) e^{i(-k_{[N]} \cdot x_{[N]} + l_{[N]} \cdot y_{[N]})} dx_{[N]} dy_{[N]}.$$

Then $F^N$ is invariant by translation if and only if

$$\sum_{j=1}^{N} k_j - l_j \neq 0 \Rightarrow \hat{F}_N(k_{[N]}, l_{[N]}) = 0.$$

Note that if $F_N$ is invariant by translation, then for any $\omega \subset [N]$, $\mathrm{tr}_\omega F^N$ is also invariant by translation.

As the Hamiltonian of the system commutes with all the translation, $F^N(t)$ remains invariant by translation if the initial data is. In addition the function $x \mapsto f_1^N(t,x,x)$ is constant and so the effective potential has no effect.

We can now compute the equations verified by $f_1^N$.

$$i\hbar\partial_t F_N^{(1)} = [-\tfrac{\hbar^2}{2}\Delta, F_N^{[1]}] + \frac{N-1}{N}\operatorname{tr}_1[V_{1,2}, F_N^{[2]}] = [-\tfrac{\hbar^2}{2}\Delta, F_N^{[1]}] + \frac{N-1}{N}\operatorname{tr}_1[V_{1,2}, G_N^{[2]}]. \tag{2.7}$$

using that for denoting $H^{(n)}(X_n,Y_n)$ the kernel of $H^{(n)} \in \mathcal{L}^1(\mathfrak{H}_L^n)$, for $F \in \mathcal{L}^1(\mathfrak{H}_L)$ invariant by translation, and $\mathcal{V}$ has spherical symmetry, $F(x,y) = \tilde{F}(x-y)$ for some $\tilde{F}$ and

$$\operatorname{tr}_1[V_{1,2}, F\otimes F](x,y) = \int (\mathcal{V}(x-z) - \mathcal{V}(y-z))\tilde{F}(x-y)\tilde{F}(z-z)\,\mathrm{d}z = 0.$$

To compute the equation verified by $G_N^{(2)}(t)$, we begin by derivating $F_N^{(2)}$. We denote

$$\begin{aligned}
i\hbar\partial_t F_N^{[2]} =& [\tfrac{\hbar^2}{2}(-\Delta_1 - \Delta_2) + \frac{1}{N}V_{1,2}, F_N^{[2]}] + \frac{N-2}{N}\operatorname{tr}_{12}[V_{1,3}+V_{2,3}, F_N^{[3]}] \\
=& [\tfrac{\hbar^2}{2}(-\Delta_1 - \Delta_2) + \frac{1}{N}V_{1,2}, G_N^{[2]}] + F_N^{\{1\}}[-\tfrac{\hbar^2}{2}\Delta_2, F_N^{\{2\}}] + [-\tfrac{\hbar^2}{2}\Delta_1, F_N^{\{1\}}]F_N^{\{2\}} \\
&+ \frac{1}{N}[V_{1,2}, F_N^{\{1\}}F_N^{\{2\}}] + \frac{N-2}{N}\Big(\operatorname{tr}_{12}[V_{2,3}, G_N^{\{1,2\}}F_N^{\{3\}}] + \operatorname{tr}_{12}[V_{1,3}, G_N^{\{1,2\}}F_N^{\{3\}}]\Big) \\
&+ \frac{N-2}{N}\Big(F_N^{\{1\}}\operatorname{tr}_2\Big[V_{2,3}, F_N^{\{2\}}F_N^{\{3\}} + G_N^{\{2,3\}}\Big] + \operatorname{tr}_2\Big[V_{1,3}, F_N^{\{1\}}F_N^{\{3\}} + G_N^{\{1,3\}}\Big]F_N^{\{2\}}\Big) \\
&+ \frac{N-2}{N}\Big(\operatorname{tr}_{12}[V_{2,3}, G_N^{\{1,3\}}F_N^{\{2\}}] + \operatorname{tr}_{12}[V_{1,3}, G_N^{\{2,3\}}F_N^{\{1\}}] + \operatorname{tr}_{12}[V_{1,3}+V_{2,3}, G_N^{\{1,2,3\}}]\Big).
\end{aligned}$$

Using that $\operatorname{tr}_{12}[V_{2,3}, G_N^{(2)}(12)F_N^{(1)}(3)]$ and $\operatorname{tr}_{12}[V_{1,3}, G_N^{(2)}(12)F_N^{(1)}(3)]$ vanish, (in the same way that )

$$\begin{aligned}
i\hbar\partial_t G_N^{\{1,2\}} =& \Big[\tfrac{\hbar^2}{2}(-\Delta_1 - \Delta_2) + \frac{1}{N}V_{1,2}, G_N^{\{1,2\}}\Big] + \frac{1}{N}\Big[V_{1,2}, F_N^{\{1\}}F_N^{\{2\}}\Big] + \frac{N-2}{N}\Big(\operatorname{tr}_{12}\Big[V_{1,3}, g_{13}^N f_2^N\Big] \\
&+\operatorname{tr}_{12}\Big[V_{2,3}, G_N^{\{1,3\}}F_N^{(2)}\Big] + \operatorname{tr}_{12}\Big[V_{1,3}+V_{2,3}, G_N^{\{1,2,3\}}\Big]\Big) - \frac{1}{N}\Big(F_N^{\{1\}}\operatorname{tr}_2\Big[V_{2,3}, G_N^{\{2,3\}}\Big] + \operatorname{tr}_1\Big[V_{1,3}, G_N^{\{1,3\}}\Big]F_N^{\{2\}}\Big).
\end{aligned}$$

Consider a family of initial data $F_0^N \in \mathcal{L}_s^1(\mathfrak{H}_L^N)$ invariant by translation and such that the associated families $(G_N^{[n]}(t))_{n\le N}$ vanish at time 0. Then for $\frac{t}{\hbar} \le \frac{\beta \log N}{C\|\mathcal{V}\|_\infty}$, $F_N^{[1]}$ and $G_N^{[2]}$ verify

$$i\hbar N\partial_t F_N^{[1]} = \operatorname{tr}_{[1]}[V_{1,2}, NG_N^{[2]}] + O_{\mathcal{L}^1}\left(N^{-1+\beta}\right), \tag{2.8}$$

$$\begin{aligned}
i\hbar\partial_t NG_N^{[2]} =& [\hbar^2(-\Delta_1 - \Delta_2), NG_N^{[2]}] + \operatorname{tr}_{[2]}\Big(\Big[V_{1,3}, F_N^{\{1\}}G_N^{\{2,3\}}\Big] + \Big[V_{2,3}, F_N^{\{2\}}G_N^{\{1,3\}}\Big]\Big) \\
&+ [V_{1,2}, F_N^{\{1\}}F_N^{\{2\}}] + O_{\mathcal{L}^1}\Big(N^{\frac{-1+9\beta}{2}}\Big).
\end{aligned} \tag{2.9}$$

We define $F_N^L : \mathbb{R}^+ \to \mathcal{L}_s^1(\mathfrak{H}_L)$ and $G_N^L : \mathbb{R}^+ \to \mathcal{L}_s^1(\mathfrak{H}_L^2)$ the solutions of the equation

$$\left\{\begin{aligned}
&F_0^L := F_N^L(t=0) = F_1^N(0),\ G_0^L := 0, \\
&i\hbar N\partial_t F_N^L = \operatorname{tr}_1[V_{1,2}, G_N^L], \\
&i\hbar\partial_t G_N^L = [\tfrac{\hbar^2}{2}(-\Delta_1 - \Delta_2), G_N^L] + \operatorname{tr}_{[2]}\Big(\Big[V_{1,3}, F_N^{L,\{1\}}G_N^{L,\{2,3\}}\Big] + \Big[V_{2,3}, F_N^{L,\{2\}}G_N^{L,\{1,3\}}\Big]\Big) \\
&\qquad + [V_{1,2}, F_0^L F_0^L].
\end{aligned}\right. \tag{2.10}$$

From a straight forward application Proposition 2.1 and Gronwall Lemma, we obtain

**Proposition 2.3.** *For* $\frac{t}{\hbar} \le \frac{\beta \log N}{C\|\mathcal{V}\|_\infty}$,

$$\|i\hbar N\partial_t(F_N^L(t) - F_N^{[1]}(t))\| + \|G_N^L(t) - NG_N^{[2]}(t)\| \le O\left(N^{\frac{-1+9\beta}{2}}\right). \tag{2.11}$$

## 2.3. **The Larg Box limit.**

In this section we want to look at the limit $L \to \infty$.

We begin by writing the equation (2.10) in the Fourier variables. As the matrix $F_N^L$ and $G_N^L$ are invariant by translation,

$$\hat{F}_N^L(k_1,l_1) = 0 \text{ if } k_1 \neq l_1,\ \hat{G}_N^L(k_1,k_2,l_1,l_2) = 0 \text{ if } k_1+k_2 \neq l_1+l_2.$$

We introduce

$$\forall v_1 \in \mathbb{Z}_{L/\hbar},\ f_{N,L}(t,v_1) := \frac{L^d}{(2\pi\hbar)^d} \hat{F}^L_{\hbar,N}(t, \tfrac{v_1}{\hbar}, -\tfrac{v_1}{\hbar}), \tag{2.12}$$

$$g_{N,L}(t,v_1,v_2,k) := \frac{L^{2d}}{(2\pi\hbar)^{2d}} \hat{G}^L_N(t, \tfrac{v_1}{\hbar}, \tfrac{v_2}{\hbar}, \tfrac{v_1}{\hbar}+k, \tfrac{v_2}{\hbar}-k). \tag{2.13}$$

Using that $\hat{F}^L_N$ and $\hat{G}^L_N$ are moreover Hermitian and positive, one have

$$f_{N,L}(t,v_1) \in \mathbb{R}^+,\ g_{N,L}(t,v_1,v_2,k) = \overline{g_{N,L}(t,v_1+\hbar k, v_2-\hbar k, -k)}. \tag{2.14}$$

With the change of unknown, the initial data becomes

$$\forall v_1 \in \mathbb{Z}_{L/\hbar},\ f_{0,L}(v_1) = \frac{1}{\mathcal{Z}_L} f_0(v_1). \tag{2.15}$$

**Proposition 2.4.** *Denoting for any function $\Delta_k h(v) = h(v) - h(v-\hbar k)$, and $E(v) := \frac{|v|^2}{2}$,*

$$\begin{cases} \hbar N i \partial_t f_{N,L}(v_1) = -2\dfrac{(2\pi\hbar)^d}{L^{2d}} \displaystyle\sum_{k\in\mathbb{Z}_L,\, v_2\in\mathbb{Z}_{L/\hbar}} \hat{\mathcal{V}}(k)(g_{N,L}(v_1,v_2,k) - g_{N,L}(v_1+\hbar k, v_2-\hbar k,-k)), \\ \hbar\partial_t g_{N,L}(t) + i(K_2+B^L) g_{N,L}(t) = iA^L \\ A^L(v_1,v_2,k) := \hat{\mathcal{V}}(k)(f_{0,L}(v_1)f_{0,L}(v_2) - f_{0,L}(v_1+\hbar k) f_{0,L}(v_2-\hbar k)) \end{cases} \tag{2.16}$$

*where $K_2$ is the kinetic part, an unbounded multiplicative operator*

$$K_2 g(v_1,v_2,k) := \big(\Delta_{-k}E(v_1) + \Delta_k E(v_2)\big) g(v_1,v_2,k) \tag{2.17}$$

*and $B^L$ is a bounded operator on (note that it is even compact)*

$$B^L g(v_1,v_2,k) := \hat{\mathcal{V}}(k)\left(\Delta_{-k} f_{0,L}(v_1) \frac{(2\pi)^d}{\hbar^d L^d} \sum_{\tilde{v}_1\in\mathbb{Z}_{L/\hbar}} g(\tilde{v}_1,v_2,k) + \Delta_k f_{0,L}(v_2) \frac{(2\pi)^d}{\hbar^d L^d} \sum_{\tilde{v}_2\in\mathbb{Z}_{L/\hbar}} g(v_1,\tilde{v}_2,k)\right) \tag{2.18}$$

*Proof.* We recall that

$$i\hbar N \partial_t F^L_N = \mathrm{tr}_1[V_{1,2}, G^L_N],$$

$$i\hbar\partial_t G^L_N = [\tfrac{\hbar^2}{2}(-\Delta_1 - \Delta_2), G^L_N] + \mathrm{tr}_{[2]}\left(\left[V_{1,3}, F_N^{L,\{1\}} G_N^{L,\{2,3\}}\right] + \left[V_{2,3}, F_N^{L,\{2\}} G_N^{L,\{1,3\}}\right]\right) + [V_{1,2}, F^L_0 F^L_0].$$

As the computation of $\partial_t f^L_N$ and $\partial_t g^L_N$ are similar, we will only compute the first one.
We set $\Psi := [V_{1,2}, G^L_N]$.

$$\begin{aligned} \Psi(x,y) &= \int [\mathcal{V}(x-z) - \mathcal{V}(z-y)]\, G^L_N(x,z,y,z)\,\mathrm{d}z \\ &= \frac{1}{L^{5d}} \sum_{k,\ell_1,\ell_2,p_1,p_2\in\mathbb{Z}_L} \int \left[e^{i(k(x-z)+\ell_1 x+\ell_2 z - p_1 y - p_2 z)} - e^{i(k(z-y)+\ell_1 x + \ell_2 z - p_1 y - p_2 z)}\right] \hat{\mathcal{V}}(k) \hat{G}^L_N(\ell_1,\ell_2,p_1,q_2)\,\mathrm{d}z \\ &= \frac{1}{L^{4d}} \sum_{k,\ell_1,\ell_2,p_1\in\mathbb{Z}_L} \left[e^{i((k+\ell_1)x - p_1 y)} \hat{G}^L_N(\ell_1,\ell_2,p_1,\ell_2-k) - e^{i(\ell_1 x - (k+p_1)y)} \hat{G}^L_N(\ell_1,\ell_2,p_1,p_1+k)\right] \hat{\mathcal{V}}(k) \\ &= \frac{1}{L^{4d}} \sum_{k,\ell_1,\ell_2,p_1\in\mathbb{Z}_L} \left[\hat{G}^L_N(\ell_1-k,\ell_2,p_1,\ell_2-k) - \hat{G}^L_N(\ell_1,\ell_2,p_1-k,\ell_2+k)\right] \hat{\mathcal{V}}(k) e^{i(\ell_1 x - p_1 y)}. \end{aligned}$$

We deduce that

$$\hat{\Psi}(\ell_1,\ell_1) = \frac{1}{L^{2d}} \sum_{k,\ell_2\in\mathbb{Z}_L} \left[\hat{G}^L_N(\ell_1+k,\ell_2-k,\ell_1,\ell_2) - \hat{G}^L_N(\ell_1,\ell_2,\ell_1+k,\ell_2-k)\right] \hat{\mathcal{V}}(k)$$

This gives the evolution on $f_{N,L}$. □

We define the couple $(f_N, g_N)$ on $\mathbb{R}^d$ and $\mathbb{R}^{3d}$ as the solution of the equation

$$\begin{cases} \hbar N \partial_t f_N(v_1) = -\dfrac{4}{(2\pi)^d} \Im \displaystyle\int dk\, dv_2 \hat{\mathcal{V}}(k) g_N(v_1,v_2,k), \\ \hbar\partial_t g_N(t) + i(K_2+B) g_N(t) = iA \\ f_N(t=0) = \Phi_0,\ g_N(t=0) = 0 \end{cases} \tag{2.19}$$

where we denote

$$A(v_1,v_2,k) := \hat{\mathcal{V}}(k)(f_0(v_1)f_0(v_2) - f_0(v_1+\hbar k)f_0(v_2-\hbar k)) \tag{2.20}$$

$$Bh(v_1,v_2,k) := \hat{\mathcal{V}}(k)\left(\Delta_{-k}f_0(k_1)\int \mathrm{d}\tilde{v}_1 h(\tilde{v}_1,v_2,k) + \Delta_k f_0(v_2)\int d\tilde{v}_1 h(v_1,\tilde{v}_2,k)\right) \tag{2.21}$$

We want to compare the $(f_{N,L}(t), g_{N,L}(t))$ with $(f_N(t), g_N(t))$. The space $\mathcal{L}^1(\mathfrak{H}_L)$ and $\mathcal{L}^1(\mathfrak{H}_L^2)$ can be injected respectively in $\ell^1_{v_1}(\mathbb{Z}_{L/\hbar})$ and $\ell^\infty_k(\mathbb{Z}_L, \ell^1_{v_1,v_2}(\mathbb{Z}^2_{L/\hbar}))$ (see Section 1.5 of [Gol13]):

$$\frac{(2\pi\hbar)^d}{L^d}\sum_{v_1\in\mathbb{Z}_{L/\hbar}}|f(v_1)| \le \|F\|_{\mathcal{L}^1_{s,i}(\mathfrak{H}_L)} \quad\text{and}\quad \sup_{k\in\mathbb{Z}_L}\frac{(2\pi\hbar)^{2d}}{L^{2d}}\sum_{v_1,v_2\in\mathbb{Z}_{L/\hbar}}|\hat{g}(v_1,v_2,k)| \le \|G\|_{\mathcal{L}^1_{s,i}(\mathfrak{H}^2_L)}.$$

We want to find two Banach spaces, $E_1$ containing both $L^1_{v_1}(\mathbb{R}^d)$ and the $\ell^1_{v_1}(\mathbb{Z}_{L/\hbar})$ for any $L$, and $E_2$ containing both $L^\infty_k(L^1_{v_1,v_2}(\mathbb{R}^{3d}))$ and the $\ell^\infty_k(\mathbb{Z}_L, \ell^1_{v_1,v_2}(\mathbb{Z}^2_{L/\hbar})$ for any $L$. For $E_1$ we take the set of bounded radon measure on $\mathbb{R}^d$ with total variation norm, with injection

$$f\in L^1(\mathbb{R}^d)\mapsto f(v_1)\,\mathrm{d}v_1, \quad f^L\in\ell^1(\mathbb{Z}_L)\mapsto \frac{(2\pi\hbar)^d}{L^d}\sum_{v_1\in\mathbb{Z}_{L/\hbar}} f^L(v_1)\delta_{v_1}.$$

For $E_2$ we consider the space of Radon measure $h$ on $\mathbb{R}^{3d}$ such that the norm

$$\|g\|_{E_2} := \sup_{\substack{k\in\mathbb{R}^d\\ \varepsilon>0}}\frac{1}{|B_k(\varepsilon)|}\|g(v_1,v_2,\tilde{k})\|_{TV((B_k(\varepsilon))_k\times\mathbb{R}^d_{v_1}\times\mathbb{R}^d_{v_2})}$$

is finite, with the injection

$$g\in L^\infty_k(L^1_{v_1,v_2})\mapsto g(k,v_1,v_2)\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2,$$

$$g^L\in\ell^\infty_k(\ell^1_{v_1,v_2}(\mathbb{Z}^3_L)\mapsto\frac{(2\pi)^{3d}\hbar^{2d}}{L^{3d}}\sum_{\substack{v_1,v_2\in\mathbb{Z}_{L/\hbar},\\ k\in\mathbb{Z}_L}} g^L(v_1,v_2,k)\delta_{v_1,v_2,k}.$$

Denoting, for $h(v_1,v_2,k)$ a radon measure on $\mathbb{R}^{3d}$, (here the brackets designate the duality product)

$$\langle h(v_1,v_2,k)\rangle_{v_1} : (v_2,k)\mapsto\langle h(v_1,v_2,k), 1_{v_1}\rangle_{v_1}.$$

One can now extend the operator $B$ and $B^L$ on $E^2$ by

$$B^L h(v_1,v_2,k) := \hat{\mathcal{V}}(k)\Big(\Delta_{-k}f_0^L(v_1)\langle h(\tilde{v}_1,v_2,k)\rangle_{\tilde{v_1}} + \Delta_k f_0^L(v_2)\langle h(v_1,\tilde{v}_2,k)\rangle_{\tilde{v_2}}\Big)$$

$$Bh(v_1,v_2,k) := \hat{\mathcal{V}}(k)\Big(\Delta_{-k}f_0(v_1)\langle h(\tilde{v}_1,v_2,k)\rangle_{\tilde{v_1}} + \Delta_k f_0(v_2)\langle h(v_1,\tilde{v}_2,k)\rangle_{\tilde{v_2}}\Big)$$

**Proposition 2.5.** *Fix an initial data $f_0$ with $\|\langle v\rangle^{d+1}\langle\nabla\rangle f_0\|_{L^\infty}<\infty$ .*
*For any parameter $N,\hbar$ and $L$,*

$$N\hbar\|\partial_t(f_{N,L}-f_N)\|_{TV}\lesssim C\|g\|_{E_2}\le\frac{\|\langle v\rangle^{d+1}\langle\nabla\rangle f_0\|_{L^\infty}}{L/\hbar}e^{\frac{4\|\hat{\mathcal{V}}\|_{L^\infty}t}{\hbar}}.$$

*Proof.* We begin by the following lemma which can be directly deduced from the definition of the operator $B$ and $B^L$.

**Lemma 2.6.** *The operator $B$ and $B^L$ are bounded in $\mathcal{L}(E_2)$, with the following bound*

$$\|B\|\le\frac{4}{(2\pi)^d}\|\hat{\mathcal{V}}\|_{L^\infty}\|f_0\|_{TV},\ \|B^L\|\le\frac{4}{(2\pi)^d}\|\hat{\mathcal{V}}\|_{L^\infty}\|f_0^L\|_{TV},$$

$$\|B-B^L\|\le\frac{4}{(2\pi)^d}\|\hat{\mathcal{V}}\|_{L^\infty}\|f_0-f_0^L\|_{TV}.$$

We have

$$\partial_t(g_{N,L}-g_N)+iK_2(g_{N,L}-g_N) = -iB^L(g_{N,L}-g_L)-i(B-B^L)g_N+i(A-A^L).$$

Using that $iK_2$ is the generator of an unitary group on $E_2$ and that $\|f\|_{TV}=\|f^L\|_{TV}=1$ (we begin with two probability measure),

$$\|A-A^L\|_{E_2}\le 4\|\hat{\mathcal{V}}\|_{L^\infty}\|f_0-f_0^L\|_{TV},$$

we have by a Gronwall lemma

$$\|g_{N,L}(t) - g_N(t)\|_{E_2} \leq \|f_0 - f_0^L\|_{TV} \exp\left(\frac{4\|V\|_{L^1} t}{(2\pi)^d \hbar}\right).$$

Because $\mathcal{V}$ verifies Assumption 2,

$$N\hbar \|\partial_t (f_{N,L} - f_N)\|_{TV} \lesssim \|g\|_{E_2} \lesssim \|f_0 - f_0^L\|_{TV} \exp\left(\frac{4\|\mathcal{V}\|_{L^1} t}{(2\pi)^d \hbar}\right).$$

We need to $\|f_0 - f_0^L\|_{TV}$, We can decompose $\mathcal{Z}_L$ as

$$\mathcal{Z}_L = 1 + \frac{(2\pi\hbar)^d}{L^d} \sum_{v \in \mathbb{Z}_{L/\hbar}} \left( f_0(v) - \int_{[-\frac{1}{2},\frac{1}{2}]^d} f_0(v + \tfrac{2\pi}{L/\hbar}\ell) d\ell \right)$$

hence

$$\begin{aligned}\frac{(2\pi\hbar)^d}{L^d} &\sum_{v \in \mathbb{Z}_{L/\hbar}} \left| \frac{f_0(v)}{\mathcal{Z}_L} - \int_{[-\frac{1}{2},\frac{1}{2}]^d} f_0(v + \tfrac{2\pi}{L/\hbar} w)\, \mathrm{d}w \right| \\ &\leq \left| 1 - \frac{1}{\mathcal{Z}_L} \right| + \frac{(2\pi\hbar)^d}{L^d} \sum_{v \in \mathbb{Z}_{L/\hbar}} \left| f_0(v) - \int_{[-\frac{1}{2},\frac{1}{2}]^d} f_0(v + \tfrac{2\pi}{L/\hbar} w)\, \mathrm{d}w \right| \\ &\lesssim \frac{\hbar^d}{L^d} \sum_{v \in \mathbb{Z}_{L/\hbar}} \frac{\|\langle x \rangle^{d+1} \langle \nabla \rangle f_0\|_{L^\infty}}{L/\hbar \langle v \rangle^{d+1}} \lesssim \frac{\|\langle x \rangle^{d+1} \langle \nabla \rangle f_0\|_{L^\infty}}{L/\hbar}.\end{aligned}$$

□

2.4. **Computation of the collision term.** In this final section, we study the limit of the $(f_N, g_N)$. The first step is to apply dome Laplace transform in time

**Proposition 2.7.** *For any smooth test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,*
(2.22)

$$N \int_0^\infty \left\langle \partial_t \tilde{f}_N(\tau_N t, v_1) \varphi(t, v_1) \right\rangle_{v_1} \mathrm{d}t = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{i\pi\hbar} \left\langle \frac{\tilde{\varphi}(\alpha, v_1)}{1 + i\alpha} \left( T\left(\tfrac{\hbar(i-\alpha)}{\tau_N}, k, v_1\right) - T\left(\tfrac{\hbar(i-\alpha)}{\tau_N}, -k, v_1'\right) \right) \right\rangle_{k, v_1}$$

*where $\tilde{\varphi}(\alpha, v_1)$ is the Laplace transform in time of $\varphi$.*

$$\tilde{\varphi}(\alpha, v_1) := \frac{1}{2\pi} \int_0^\infty e^{t(1+i\alpha)} \varphi(t, v_1)\, \mathrm{d}t$$

*and*

$$T(\omega, k, v_1) := \frac{\hat{\mathcal{V}}(k)}{(2\pi)^d} \left\langle \left((K_2 + B) - \omega\right)^{-1} A(v_1, v_2, k) \right\rangle_{v_2}. \tag{2.23}$$

Note that the preceding proposition is justified by the existence of the resolvent $\left(K_2 + B - \frac{\hbar}{\tau_N}(i - \alpha)\right)^{-1}$.

*Proof.* The proof is the same than Lemma 5.2 of [DSR21].
We recall that $g_N$ is solution of

$$\begin{gathered}\hbar \partial_t g_N(t) + i(K_2 + B) g_N(t) = iA \\ g_N(t = 0) = 0.\end{gathered}$$

Using the Duhamel form of the evolution equation of $g_N$, and that $\delta(t) = \frac{1}{2\pi} \int e^{i\alpha t}\, \mathrm{d}\alpha$,

$$\begin{aligned}\int_0^\infty g_N(\tau_N t) \varphi(t)\, \mathrm{d}t &= \frac{i}{\hbar} \int_0^\infty \int_0^{\tau_N t} e^{-\frac{i}{\hbar}(K_2+B)(t-t_1)} A\, \mathrm{d}t_1 \varphi(t, v_1)\, \mathrm{d}t \\ &= \frac{i}{\hbar} \int_{(\mathbb{R}^+)^3} e^{-(\frac{1}{\tau_N} + \frac{i}{\hbar}(K_2+B))t_2} A e^{-\frac{t_2}{\tau_N}} e^t \varphi(t) \delta_{t_1+t_2=\tau_N t}\, \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t \\ &= \frac{i}{2\pi\hbar\tau_N} \int_{\mathbb{R}} \int_{(\mathbb{R}^+)^3} e^{-(\frac{1+i\alpha}{\tau_N} + \frac{i}{\hbar}(K_2+B))t_2} A e^{-\frac{1+i\alpha}{\hbar\tau_N} t_1} e^{(1+i\alpha)t} \varphi(t)\, \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t\, \mathrm{d}\alpha \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\hbar}{\tau_N}(\alpha - i) + (K_2 + B) \right)^{-1} A \;\; \frac{\tilde{\varphi}(\alpha)}{1 + i\alpha}\, \mathrm{d}\alpha\end{aligned}$$

Using the symmetry property of $g_N(t, v_1, v_2, k)$, one has

$$N\partial_t f_N(v_1) = -\frac{2}{i\hbar} \left\langle \hat{\mathcal{V}}(k) \left( g_N(v_1, v_2, k) - g_N(v_1', v_2', -k) \right) \right\rangle_{v_2,k},$$

where $v_1' := v_1 + \hbar k$ and $v_2' := v_2 - \hbar k$.

The result is obtain by the combination of the both identities.

□

**Proposition 2.8.** *The function $T(\omega, k, v_1)$ verifies the following boundness and convergence results: for*

$$|T(\omega, k, v_1) - T(\omega, -k, v_1')| \lesssim \hat{\mathcal{V}}^2(k) \|\langle v\rangle^2 \langle\nabla\rangle^2 f_0\|_{L^\infty}^2 \log\left(1 + \frac{|\Re\omega|}{|\Im\omega|}\right)^2, \tag{2.24}$$

$$\lim_{\substack{\omega\to 0\\ \Im\omega > |\Re\omega|}} \frac{T(\omega, k, v_1) - T(\omega, -k, v_1')}{2\pi i} = \frac{\hat{\mathcal{V}}^2(k) \left\langle (f_0(v_2') f_0(v_1') - f_0(v_1) f_0(v_2)) \delta_{\Delta_k E(v_2) + \Delta_{-k} E(v_1)} \right\rangle_{v_2}}{(2\pi)^d |\tilde{\varepsilon}\left(k, i0 - \Delta_{-k} E(v_1)\right)|^2}, \tag{2.25}$$

*Proof.* We decompose $(K_2 + B) = L_1 + L_2$, where $L_1$,$L_2$ are defined by

$$L_1 h(v_1, v_2, k) := \Delta_{-k} E(v_1) h(v_1, v_2, k) + \hat{\mathcal{V}}(k) \Delta_{-k} f_0(v_1) \left\langle h(\tilde{v}_1, v_2, k)\right\rangle_{\tilde{k}_1} \tag{2.26}$$

$$L_2 h(v_1, v_2, k) := \Delta_k E(v_2) h(v_1, v_2, k) + \hat{\mathcal{V}}(k) \Delta_k f_0(v_2) \left\langle h(v_1, \tilde{v}_2, k)\right\rangle_{\tilde{k}_2}. \tag{2.27}$$

The operators $L_1$ and $L_2$ are easier to study than $K_2 + B$. We have

$$((K_2 + B^L) - \omega)^{-1} = \frac{1}{2\pi i} \int (L_2 + \beta - \frac{\omega}{2})^{-1} (L_2 - \beta - \frac{\omega}{2})^{-1} \mathrm{d}\beta \tag{2.28}$$

The resolvant $L_1$ and $L_2$ are given by

$$(L_1 - \omega)^{-1} h = \frac{h}{\Delta_{-k} E(v_1) - \omega} - \frac{\hat{\mathcal{V}}(k)}{\tilde{\varepsilon}(-k, \omega)} \frac{\Delta_{-k} f_0(v_1)}{\Delta_{-k} E(v_1) - \omega} \left\langle \frac{h}{\Delta_{-k} E(\tilde{v}_1) - \omega} \right\rangle_{\tilde{v}_1} \tag{2.29}$$

$$(L_2 - \omega)^{-1} h = \frac{h}{\Delta_k E(v_2) - \omega} - \frac{\hat{\mathcal{V}}(k)}{\tilde{\varepsilon}(k, \omega)} \frac{\Delta_k f_0(v_2)}{\Delta_k E(v_2) - \omega} \left\langle \frac{h}{\Delta_k E(\tilde{v}_2) - \omega} \right\rangle_{\tilde{v}_2}. \tag{2.30}$$

where the (modified) dielectric constant $\tilde{\varepsilon}$ is defined by

$$\tilde{\varepsilon}(k, \omega) := 1 + \hat{\mathcal{V}}(k) \left\langle \frac{\Delta_k f_0(k_*)}{\Delta_k E(v_*) - \omega} \right\rangle_{v_*} = 1 + \hat{\mathcal{V}}(k) \int \frac{f_0(k_*) - f_0(v_* - \hbar k)}{\hbar k \cdot \left(v_* - \frac{\hbar k}{2}\right) - \omega} \mathrm{d}k_* \tag{2.31}$$

First we give an estimation of $\tilde{\varepsilon}$

**Lemma 2.9.** *There exists a constant $C > 0$ such that if $f_0$ satifies the bound $\|\mathcal{V}\|_{L^1} \|\langle v\rangle^{2d} \langle\nabla\rangle^2 f_0\|_{L^\infty} < 1/C$, the function $\omega \mapsto \tilde{\varepsilon}(k, \omega)$ is holomorphic on $\{\omega, \Im\omega \neq 0\}$ and there exist a constant $\eta \in (0, 1)$ independent of $\omega$ such that $|\tilde{\varepsilon}(k, \omega)| > \eta$. In addition,*

$$|1 - \tilde{\varepsilon}(k, \omega)| \lesssim \begin{cases} \|\langle v\rangle^{2d} \langle\nabla\rangle^2 f_0\|_{L^\infty} / |\Im\omega|, \\ \|\langle v\rangle^{2d} \langle\nabla\rangle^2 f_0\|_{L^\infty} / |\Re\omega| \text{ if } |\Re\omega| > 4\hbar^2 |k|^2. \end{cases} \tag{2.32}$$

*Proof.* We denote $\tilde{\omega} := \frac{\Re\omega}{\hbar^2 |k|^2}$,

$$\int \frac{f_0(v_*) - f_0(v_* - \hbar k)}{\hbar k \cdot (v_* - \frac{1}{2}\hbar k) - \omega} \mathrm{d}v_* = \int \frac{f_0(v_* + \hbar k(\frac{1}{2} + \tilde{\omega}) - f_0(v_* + \hbar k(-\frac{1}{2} + \tilde{\omega}))}{\hbar k \cdot v_* - i\Im\omega} \mathrm{d}v_*$$

Hence we can decompose it in real and imaginary parts

$$\Re \int \frac{f_0(v_*) - f_0(v_* - \hbar k)}{\hbar k \cdot (v_* - \frac{1}{2}\hbar k) - \omega} \mathrm{d}v_* = \int_{-1/2}^{1/2} \int \frac{(v_* \cdot \hbar k)\, \hbar k \cdot \nabla f\left(v_* + \hbar k\left(s + \tilde{\omega}\right)\right)}{(\hbar k \cdot v_*)^2 + (\Im\omega)^2} \mathrm{d}v_* \mathrm{d}s$$

$$\Im \int \mathrm{d}v_* \frac{f_0(v_*) - f_0(v_* - \hbar k)}{\hbar k \cdot (v_* - \frac{1}{2}\hbar k) - \omega} = \int_{-1/2}^{1/2} \mathrm{d}s \int \frac{\Im\omega\, \hbar k \cdot \nabla f\left(v_* + \hbar k\left(s + \tilde{\omega}\right)\right)}{(\hbar k \cdot v_*)^2 + (\Im\omega)^2} \mathrm{d}v_*$$

We only need to bound the real part of $\tilde{\varepsilon}$ to get a bound by below. Decomposing $v_*$ in orthogonally as $v_* := v_\parallel + v_\perp$, where $v_\perp \cdot k = 0$, and denoting $\tilde{v}_s := \hbar\,(s+\tilde{\omega})\,k$, we get for any constant $a > 0$

$$\begin{aligned}\Re \int \frac{f_0(v_*) - f_0(v_* - k)}{\hbar k \cdot (v_* - \frac{1}{2}\hbar k) - \omega} \mathrm{d}v_* &\leq \int_{-1/2}^{1/2} \int_{|\hbar k \cdot v_*| \geq a} \frac{\hbar |k|\, |\nabla f\,(v_* + \tilde{v}_s)|}{a} \mathrm{d}v_* \,\mathrm{d}s \\ &+ \int_{-1/2}^{1/2} \int_{|\hbar k \cdot v_\parallel| < a} \left| \frac{\hbar v_\parallel \cdot k\, \hbar k \cdot \left(\nabla f\left(v_\parallel + v_\perp + \tilde{v}_s\right) - \nabla f\left(-v_\parallel + v_\perp + \tilde{v}_s\right)\right)}{(\hbar k \cdot v_\parallel)^2 + (\Im\omega)^2} \right| \mathrm{d}v_* \,\mathrm{d}s \\ &\lesssim \left( \frac{\hbar|k|}{a} + \frac{a}{\hbar|k|} \right) \|\langle v\rangle^{2d} \langle\nabla\rangle^2 f_0\|_{L^\infty}\end{aligned}$$

Choosing $a = \hbar|k|$, for some contant $c > 0$,

$$|\tilde{\varepsilon}(k,\omega)| \geq 1 - c\|\langle v\rangle^{2d}\langle\nabla\rangle^2 f_0\|_{L^\infty} > 0.$$

The bound with respect to $|\Im\omega|^{-1}$ is direct. We suppose now that $\Re\omega > 4\hbar^2|k|$. Then $\tilde{v}_s \cdot \hat{k}$ is bigger than $\frac{\Re\omega}{4\hbar|k|}$.

$$\begin{aligned}\left| \int \frac{f_0(v_*) - f_0(v_* - k)}{\hbar k \cdot (v_* - \frac{1}{2}\hbar k) - \omega} \mathrm{d}v_* \right| &\leq \int_{-1/2}^{1/2} \int_{|\hbar k \cdot v_*| \geq \frac{\Re\omega}{4}} \frac{\hbar |k|\, |\nabla f\,(v_* + \tilde{v}_s)|}{\Re\omega/4} \mathrm{d}v_* \,\mathrm{d}s \\ &+ \int_{-1/2}^{1/2} \int_{|\hbar k \cdot v_\parallel| < \frac{\Re\omega}{4}} \left| \frac{\hbar v_\parallel \cdot k\, \hbar k \cdot \left(\nabla f\left(v_\parallel + v_\perp + \tilde{v}_s\right) - \nabla f\left(-v_\parallel + v_\perp + \tilde{v}_s\right)\right)}{(\hbar k \cdot v_\parallel)^2 + (\Im\omega)^2} \right| \mathrm{d}v_* \,\mathrm{d}s \\ &+ \int_{-1/2}^{1/2} \int_{|\hbar k \cdot v_\parallel| < \frac{\Re\omega}{4}} \left| \frac{\Im\omega\, \hbar k \cdot \left(\nabla f\left(v_\parallel + v_\perp + \tilde{v}_s\right) + \nabla f\left(-v_\parallel + v_\perp + \tilde{v}_s\right)\right)}{(\hbar k \cdot v_\parallel)^2 + (\Im\omega)^2} \right| \mathrm{d}v_* \,\mathrm{d}s \\ &\lesssim (\hbar|k| + 1) \frac{\|\langle v\rangle^2 \langle\nabla\rangle^2 f_0\|_{L^\infty}}{\Re\omega}\end{aligned}$$

where we use the estimation

$$\frac{1}{1 + |v_\parallel + v_\perp + \tilde{v}_s|^{2d}} \lesssim \frac{1}{(1 + |v_\parallel + \tilde{v}_s|^d)(1 + |v_\perp|^d)} \lesssim \frac{\hbar|k|}{\Re\omega(1 + |v_\perp|^d)},$$

$$\int_{\mathbb{R}} \frac{\hbar|k|\Im\omega\,\mathrm{d}x}{(\hbar|k|x)^2 + (\Im\omega)^2} = \pi.$$

We conclude using that $\hat{\mathcal{V}}(k) = O(\langle k\rangle^{-1})$. ☐

Fix $\beta \in \mathbb{R}$ and $\omega \in \mathbb{C} \setminus \mathbb{R}$. We introduce now the application

$$\mathcal{B}(v_1, k, \beta) := \left\langle \left( L_2 + \beta - \frac{\omega}{2} \right)^{-1} \left( L_1 - \beta - \frac{\omega}{2} \right)^{-1} A(v_1, v_2, k) \right\rangle_{v_2},$$

and we want to compute the limit as $\omega \to 0$ with $\Im\omega > |\Re\omega|$ of $\frac{1}{2\pi i}\int_{\mathbb{R}} B(v_1, k)\,\mathrm{d}\beta$. For $v_3 \in \mathbb{R}^d$, we denote $v_3' := v_3 + \hbar k$

$$\begin{aligned}\mathcal{B}(v_1, k, \beta) &= \frac{\hat{\mathcal{V}}(k)}{\tilde{\varepsilon}\left(k, \frac{\omega}{2} - \beta\right)} \left\langle \frac{f_0(v_1) f_0(v_2) - f_0(v_1') f_0(v_2')}{\left(\Delta_{-k} E(v_1) - \beta - \frac{\omega}{2}\right)\left(\Delta_k E(v_2) + \beta - \frac{\omega}{2}\right)} \right\rangle_{v_2} \\ &+ \frac{-\hat{\mathcal{V}}(k)^2}{\tilde{\varepsilon}\left(k, \frac{\omega}{2} - \beta\right)\tilde{\varepsilon}\left(-k, \frac{\omega}{2} + \beta\right)} \frac{\Delta_{-k} f_0(v_1)}{\Delta_{-k} E(v_1) - \beta - \frac{\omega}{2}} \left\langle \frac{f_0(v_3) f_0(v_2) - f_0(v_3') f_0(v_2')}{\left(\Delta_{-k} E(k_3) - \beta - \frac{\omega}{2}\right)\left(\Delta_k E(v_2) + \beta - \frac{\omega}{2}\right)} \right\rangle_{v_2, v_3}\end{aligned} \tag{2.33}$$

Using the identities

$$f_0(v_1) f_0(v_2) - f_0(v_1') f_0(v_2') = f_0(v_1)\Delta_k f_0(v_2) + \Delta_{-k} f_0(v_1) f_0(v_2')$$

$$\hat{\mathcal{V}}(k) \left\langle \frac{\Delta_k f_0(v_2)}{\Delta_k E(v_2) + \beta - \frac{\omega}{2}} \right\rangle_{v_2} = \tilde{\varepsilon}(k, \frac{\omega}{2} - \beta) - 1$$

the first line of (2.33) becomes

$$\frac{\tilde{\varepsilon}\left(k, \frac{\omega}{2} - \beta\right) - 1}{\tilde{\varepsilon}\left(k, \frac{\omega}{2} - \beta\right)} \frac{f_0(v_1)}{\Delta_{-k} E(v_1) - \beta - \frac{\omega}{2}} + \frac{\hat{\mathcal{V}}(k)}{\tilde{\varepsilon}\left(k, \frac{\omega}{2} - \beta\right)} \frac{\Delta_{-k} f_0(v_1)}{\Delta_{-k} E(v_1) - \beta - \frac{\omega}{2}} \left\langle \frac{f_0(v_2')}{\Delta_k E(v_2) + \beta - \frac{\omega}{2}} \right\rangle_{v_2}$$

and the second line $\frac{\hat{\mathcal{V}}(k)\Delta_{-k}f_0(v_1)}{\Delta_{-k}E(v_1)-\beta-\frac{\omega}{2}}$ times

$$\frac{1-\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)}{\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)}\left\langle\frac{f_0(v_3)}{\Delta_{-k}E(v_3)-\beta-\frac{\omega}{2}}\right\rangle_{v_3}+\frac{1-\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)}{\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)}\left\langle\frac{f_0(v_2')}{\Delta_k E(v_2)+\beta-\frac{\omega}{2}}\right\rangle_{v_2}.$$

Hence $\mathcal{B}(v_1,k,\beta)$ can be decompose into 3 peaces

$$\begin{aligned}
&\mathcal{B}_1(v_1,k,\beta):=\frac{\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)-1}{\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)}\frac{f_0(v_1)}{\Delta_{-k}E(v_1)-\beta-\frac{\omega}{2}}\\
&\mathcal{B}_2(v_1,k,\beta):=-\frac{\hat{\mathcal{V}}(k)\Delta_{-k}f_0(v_1)}{\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)\left(\Delta_{-k}E(v_1)-\beta-\frac{\omega}{2}\right)}\left\langle\frac{f_0(v_3)}{\Delta_{-k}E(v_3)-\beta-\frac{\omega}{2}}\right\rangle_{v_3}\\
&\mathcal{B}_3(v_1,k,\beta)\\
&:=\frac{\hat{\mathcal{V}}(k)\Delta_{-k}f_0(v_1)}{(\Delta_{-k}E(v_1)-\beta-\frac{\omega}{2})\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)}\left(\left\langle\frac{f_0(v_2')}{\Delta_k E(v_2)+\beta-\frac{\omega}{2}}\right\rangle_{v_2}-\left\langle\frac{f_0(v_2')}{\Delta_k E(v_2)+\beta+\frac{\omega}{2}}\right\rangle_{v_2}\right)
\end{aligned}$$

Note that that for any function $g$, $\Delta_k g(v_1-\hbar k)=-\Delta_{-k}g(v_1)$ and that $\tilde{\varepsilon}(k,\bar{\omega})=\overline{\tilde{\varepsilon}(k,\omega)}$. The function $\beta\mapsto\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)^{-1}$ is holomorphic and bounded (thanks to Proposition 2.9) in the half plane $\{\beta\in\mathbb{C},\ \Im\beta<\Im\frac{\omega}{2}\}$. In the same way $\beta\mapsto\tilde{\varepsilon}\left(-k,\frac{\omega}{4}+\beta\right)^{-1}$ is holomorphic in $\{\beta\in\mathbb{C},\beta,\ \Im\beta>-\Im\frac{\omega}{2}\}$. Using the residue Theorem, for any $R>\max(\Im\omega,4\hbar^2|k|^2)$ large enough,

$$\begin{aligned}
\int_{\mathbb{R}}\mathcal{B}_1(k,v_1,\beta)\frac{\mathrm{d}\beta}{2i\pi}=&-\frac{f_0(v_1)(\tilde{\varepsilon}(k,\omega-\Delta_{-k}E(v_1))-1)}{\tilde{\varepsilon}(k,\omega-\Delta_{-k}E(v_1))}\\
&+\int_{]-\infty,-R]\sqcup[-R,(-1+i)R]\sqcup[(-1+i)R,(1+i)R]\sqcup[(1+i)R,R]\sqcup[R,\infty[}B_1(k,v_1,\beta)\frac{\mathrm{d}\beta}{2i\pi}
\end{aligned}$$

Using the estimation of Proposition 2.9, the remaining integral is of order $O(R^{-1})$, and vanishes as $R\to\infty$. We can conclude that

$$\begin{aligned}
\lim_{\substack{\omega\to0\\ \Im\omega>|\Re\omega|}}\int\left(\mathcal{B}_1(k,v_1,\beta)-\mathcal{B}_1(-k,v_1',\beta)\right)\frac{\mathrm{d}\beta}{2\pi i}&=f_0(v_1)\left(\frac{1}{\tilde{\varepsilon}\left(k,i0-\Delta_{-k}E(v_1)\right)}-\frac{1}{\tilde{\varepsilon}\left(-k,i0-\Delta_k E(v_1')\right)}\right)\\
&=\frac{2if_0(v_1)}{|\tilde{\varepsilon}(k,+i0-\Delta_{-k}E(v_1))|^2}\Im\left(1+\frac{\hat{\mathcal{V}}(k)}{(2\pi)^d}\left\langle\frac{\Delta_k f(v_2)}{\hbar k\cdot(v_2-v_1-\hbar k)+i0}\right\rangle_{v_2}\right)\\
&=-\frac{2\pi i\hat{\mathcal{V}}(k)f_0(v_1)}{\tilde{\varepsilon}(k,+i0-\Delta_{-k}E(v_1))|^2}\left\langle\Delta_k f(v_2)\delta_{2\hbar k\cdot(v_2-v_1-k)}\right\rangle_{v_2}.
\end{aligned}$$

In the last line, we used the Sokhotskii-Plemelj formula

$$\Im\frac{1}{k\cdot(v_2-v_1)+i0}=-\pi\delta_{k\cdot(v_2-v_1)}.$$

In the same way, $\mathcal{B}_2(v_1,k,\beta)$ is holomorphic on the half plane $\{\beta\in\mathbb{C},\ \Im\beta<\Im\frac{\omega}{2}\}$

$$\lim_{\substack{\omega\to0\\ \Im\omega>|\Re\omega|}}\int\left(\mathcal{B}_2(k,v_1',\beta)-\mathcal{B}_2(-k,-v_1',\beta)\right)\frac{\mathrm{d}\beta}{2\pi i}=0.$$

We treat now $\mathcal{B}_3$. Note that $\mathcal{B}_3(k,v_1,\beta))-\mathcal{B}_3(k,v_1',-\beta)$ is equal to

$$\begin{aligned}
&\frac{\hat{\mathcal{V}}(k)}{\tilde{\varepsilon}\left(k,\frac{\omega}{2}-\beta\right)\tilde{\varepsilon}\left(-k,\frac{\omega}{2}+\beta\right)}\left(\frac{\Delta_{-k}f_0(v_1)}{\Delta_{-k}E(v_1)-\beta-\frac{\omega}{2}}-\frac{\Delta_{-k}f_0(v_1)}{\Delta_{-k}E(v_1)-\beta+\frac{\omega}{2}}\right)\\
&\qquad\times\left(\left\langle\frac{f_0(v_2')}{\Delta_k E(v_2)+\beta+\frac{\omega}{2}}\right\rangle_{v_2}-\left\langle\frac{f_0(v_2')}{\Delta_k E(v_2)+\beta-\frac{\omega}{2}}\right\rangle_{v_2}\right)
\end{aligned}$$

Using Lemma 2.9, we obtain the following bound

$$\left|\left\langle \frac{f_0(v_2')}{\Delta_k E(v_2)+\beta+\frac{\omega}{2}}\right\rangle_{v_2} - \left\langle \frac{f_0(v_2')}{\Delta_k E(v_2)+\beta-\frac{\omega}{2}}\right\rangle_{v_2}\right| \le \left\langle \frac{f_0(v_2')|\omega|}{|(\Delta_k E(v_2)+\beta)^2+\omega^2/4|}\right\rangle$$
$$\le C\|\langle v\rangle^2\langle\nabla\rangle^2 f_0\|_{L^\infty}\log\left(1+\frac{|\Re\omega|}{|\Im\omega|}\right)$$
$$\left|\frac{1}{2\pi i}\int (\mathcal{B}_3(v_1,k,\beta)-\mathcal{B}_3(v_1',-k))\,\mathrm{d}\beta\right| \le C\hat{\mathcal{V}}(k)\|\langle v\rangle^2\langle\nabla\rangle^2 f_0\|_{L^\infty}^2 \log\left(1+\frac{|\Re\omega|}{|\Im\omega|}\right)^2.$$

Using the Sokhotskii-Plemelj formula,

$$\Im \lim_{\substack{\omega\to 0\\ \Im\omega>|\Re\omega|}} \frac{1}{2\pi i}\int (\mathcal{B}_3(v_1,k,\beta)-\mathcal{B}_3(v_1',-k,-\beta))\,\mathrm{d}\beta$$
$$= \frac{2\pi i\hat{\mathcal{V}}(k)\Delta_{-k}f_0(v_1)}{|\tilde{\varepsilon}(k,i0-\Delta_{-k}E(v_1))|^2}\left\langle f_0(v_2-k)\delta_{\Delta_k E(v_2)+\Delta_{-k}E(v_1)}\right\rangle_{v_2}$$

Finally,

$$\lim_{\substack{\omega\to 0\\ \Im\omega>|\Re\omega|}}\left(\left\langle \left(K_2+B-\omega\right)^{-1}A(v_1,v_2,k)\right\rangle_{v_2} - \left\langle\left(K_2+B-\omega\right)^{-1}A(v_1',v_2',-k)\right\rangle_{v_2}\right)$$
$$= \frac{2\pi i\hat{\mathcal{V}}(k)}{|\tilde{\varepsilon}(k,i0-\Delta_{-k}E(v_1))|^2}\left\langle (f_0(v_2')f_0(v_1')-f_0(v_1)f_0(v_2))\delta_{\Delta_k E(v_2)+\Delta_{-k}E(v_1)}\right\rangle_{v_2},$$

and is uniformly bounded by $C\hat{\mathcal{V}}(k)\|f\|^2$.

□

We can now conclude the proof of Theorem 4:
First we have the identity

$$Q_{LB}^{\hbar}(f)(v_1) := \frac{2}{(2\pi)^d}\left\langle \frac{\hat{\mathcal{V}}(k)^2\delta_{\Delta_k E(v_2)+\Delta_{-k}E(v_1)}}{\hbar|\tilde{\varepsilon}(k,i0-\Delta_{-k}E(v_1))|^2}(f_0(v_2')f_0(v_1')-f_0(v_1)f_0(v_2))\right\rangle_{v_2,k}.$$

Using dominated convergence theorem,

$$N\int_0^\infty \langle\partial_t f_N(\tau_N t,v_1)\varphi(t,v_1)\rangle_{v_1}\mathrm{d}t \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}}\mathrm{d}\alpha\left\langle\frac{\tilde{\varphi}(\alpha,v_1)}{1+i\alpha}Q_{LB}(f_0)(v_1)\right\rangle_{v_1}$$
$$= \int_0^\infty \left\langle\varphi(t,v_1)Q_{LB}^{\hbar}(f_0)(v_1)\right\rangle_{v_1}\mathrm{d}t$$

One can now use the estimation of Propositions 2.3 and 2.5. For $\varphi:\mathbb{R}_t^+\times\mathbb{R}_{v_1}^d$ a smooth test function with compact support in time in $[0,T_*]$, and $\tau_N=(\log N)^{1-\alpha}$ for some $\alpha\in(0,1)$,

$$N\int_0^\infty \frac{(2\pi)^d}{L^d}\sum_{k\in\mathbb{Z}_L}\varphi(t,\hbar k)\,\partial_t\hat{F}_N^1(\tau_N t,k,k)\,\mathrm{d}t - N\int_0^\infty\langle\partial_t f_N(\tau_N t,v_1)\varphi(t,v_1)\rangle_{v_1}\mathrm{d}t$$
$$= O\left(\|N\partial_t(\hat{F}_N^1-\hat{F}_{N,L})\|_{L^\infty([0,T_*],\mathcal{L}^1)}\right) + O\left(\|N\partial_t(f_N-f_{N,L})\|_{L^\infty([0,T_*],E_1)}\right)$$
$$= O\left(N^{-1/4}+\frac{\exp\left(C\frac{T_*\tau_N}{\hbar}\right)}{L/\hbar}\right)$$

for some constant $C$. Fixing $L=N^\gamma$ for any $\gamma>0$, the preceding expression converges to 0.

## 3. Cauchy theory of the Quantum Lenard-Balescu equation

The next part is dedicated to the proof of Theorem 2. The proof is based on the work of Duerinckx and Winter [DW23], based on [Guo02].

3.1. **Bound of the dielectric constant** $\varepsilon_\hbar(F,k,v_1)$**.** We begin by bounding the dielectric constant $\varepsilon_\hbar$. The bounds are similar to the one of $\varepsilon_0$ provided in [DW23].

**Proposition 3.1.** *We fix the dimension $d \geq 2$. For any $\mathcal{V}$ satisfying Assumption 2,*

(1) Non-degenerency at the Maxwellian*: For all $k, v \in \mathbb{R}^d$ and $\omega \in \mathbb{C}$ with $\Im\omega > 0$,*

$$|\varepsilon_0(M,k,v)| \simeq 1. \tag{3.1}$$

(2) *Stability at fix $\hbar$: for $\Phi = M + \sqrt{M}f$ and $\tilde{\Phi} = M + \sqrt{M}\tilde{f}$,$\delta_0 \in ]0,1/2]$,*

$$|\varepsilon_\hbar(\Phi,k,v) - \varepsilon_\hbar(\tilde{\Phi},k,v)| \lesssim \left\| f - \tilde{f} \right\|_3. \tag{3.2}$$

(3) *Limit when $\hbar \to 0$: for $\Phi = M + \sqrt{M}f$,*

$$|\varepsilon_\hbar(\Phi,k,v) - \varepsilon_0(\Phi,k,v)| \lesssim \hbar \left\| f \right\|_2. \tag{3.3}$$

(4) *Boundedness: For $(r,q) \in \mathbb{N}^2$,*

$$|\nabla_k^r \nabla_v^q \varepsilon_\hbar(\Phi,k,v)| \lesssim \frac{\langle k\rangle^r \langle v\rangle^r}{|k|^r} \left(1 + \|f\|_{r+q+2}\right) \tag{3.4}$$

*Proof.* ***Step* 1.** The proof of (1) has been performed in Lemma 2.1 of [DW23].

***Step* 2.** Proof of (2). Setting $\hat{k} := \frac{k}{|k|}$,

$$\varepsilon_\hbar(\Phi,k,v) = 1 + \hat{\mathcal{V}}(k) \int_{-1}^0 \mathrm{d}s \int \frac{\hat{k}\cdot\nabla\Phi(v_* + s\hbar k)}{\hat{k}\cdot(v - v_* - \hbar k) - i0} \mathrm{d}v_* = 1 + \hat{\mathcal{V}}(k) \int_1^2 \mathrm{d}s \int \frac{\hat{k}\cdot\nabla\Phi(v_* - s\hbar k)}{\hat{k}\cdot(v - v_*) - i0} \mathrm{d}v_*$$

As $\varepsilon_\hbar(\cdot,k,v)$ is an affine operator, one can fix $\tilde{f} = 0$.
Using Sobolev's inequalities, we deduce for any $s \in [1,2]$

$$\left| \int \frac{\hat{k}\cdot\nabla(\sqrt{M}f)(v_* - s\hbar k)}{\hat{k}\cdot(v - v_*) - i0} \mathrm{d}v_* \right| \lesssim \left\| \int \frac{\hat{k}\cdot\nabla(\sqrt{M}f)(v_* - s\hbar k)}{\hat{k}\cdot(v - v_*) - i0} \mathrm{d}v_* \right\|_{H^1(\mathbb{R}\hat{k})}$$

Splitting the integral over $v_* \in <\hat{k}> \oplus <\hat{k}>^\perp$, and applying the usual bound on the Hilbert transform and trace operator,

$$\left| \int \frac{\hat{k}\cdot\nabla(\sqrt{M}f)(v_* - s\hbar k)}{\hat{k}\cdot(v - v_*) - i0} \mathrm{d}v_* \right| \lesssim \left\| \int \hat{k}\cdot\nabla(\sqrt{M}f)(v_*^\perp + \cdot)\,\mathrm{d}v_*^\perp \right\|_{H^1(\mathbb{R}\hat{k})} \lesssim \|f\|_2$$

***Step* 3.** Proof of (3). We have

$$\varepsilon_\hbar(\Phi,k,v) - \varepsilon_0(\Phi,k,v) = \hat{\mathcal{V}}(k) \int_1^2 \mathrm{d}s \int \frac{\hat{k}\cdot\nabla\left[F(v_* - s\hbar k) - F(v_*)\right]}{\hat{k}\cdot(v - v_*) - i0} \mathrm{d}v_*.$$

Using the same estimation as Step 2,

$$\begin{aligned} |\varepsilon_\hbar(\Phi,k,v) - \varepsilon_0(\Phi,k,v)| \lesssim& |\hat{\mathcal{V}}(k)| \int_1^2 \mathrm{d}s \left\| \langle v\rangle^{-r} \langle\nabla\rangle^{\frac{3}{2}+\delta_0} \left[ \left(M + \sqrt{M}f\right)(\cdot - s\hbar k) - \left(M + \sqrt{M}f\right) \right] \right\|_{L^2} \\ \lesssim& \hbar\left(1 + \|f\|_2\right). \end{aligned}$$

***Step* 4.** Proof of 3.4. We begin by the derivative in $v$:

$$\nabla_v^q \varepsilon_\hbar(\Phi,k,v) = \hat{\mathcal{V}}(k) \int_1^2 \mathrm{d}s \int \frac{\hat{k}\cdot\nabla^q\nabla\Phi(v - v_* - s\hbar k)}{\hat{k}\cdot v_* - i0} \mathrm{d}v_*.$$

Hence, without lost of generality, we can suppose that $q = 0$.
We apply the Leibniz formula to

$$\nabla_k^r \frac{k\cdot\nabla^q\Phi(v - v_* - s\hbar k)}{k\cdot v_* - i0}$$

For $r_1 + r_2 = r$

$$\nabla_k^{r_1} k\cdot\nabla^{q+1}\Phi(v - v_* - s\hbar k) = (-s\hbar)^{r_1} k\cdot\nabla^{q+r_1+1}\Phi(v - v_* - s\hbar k) + (-s\hbar)^{r_1-1}\nabla^{q+r}\Phi(v - v_* - s\hbar k)$$

We denote $v_*^{\parallel} := (\hat{k}\cdot v_*)\hat{k}$

$$\nabla_k^{r_2}\frac{1}{k\cdot v_* - i0} = \sum_{r_3=0}^{r_2}\binom{r_2}{r_3}\frac{(-1)^{r_2}r_2!}{|k|^{r_2-r_3}}\frac{\hat{k}^{\otimes(r_2-r_3)}\otimes_{\text{sym}}(v_*^{\perp})^{\otimes r_3}}{(k\cdot v_*-i0)^{r_3+1}}$$
$$= \frac{r_2!}{|k|^{r_2}}\sum_{r_3=0}^{k}\binom{r_2}{r_3}\frac{(-1)^{r_3}}{(r_2-r_3)!}\left[\hat{k}^{\otimes r3} : \nabla_{v_*}^{r_3}\frac{1}{k\cdot v_*-i0}\right]\hat{k}^{\otimes r_2-r_3}\otimes_{\text{sym}}(v_*^{\perp})^{\otimes r_3}$$

Using integration by part, one can transfer the $r_3$ derivative in $v_*$ into $F$.

$$\int \nabla_{v_*}^{r_3}\frac{1}{k\cdot v_*-i0}(v_*^{\perp})^{\otimes r_3}\nabla^{q+r_1}\Phi(v-v_*-s\hbar k)\,\mathrm{d}v_* = \int\frac{(v_*^{\perp})^{\otimes r_3}\nabla^{q+r_1+r_3}\Phi(v-v_*-s\hbar k)}{k\cdot v_*-i0}\mathrm{d}v_*$$
$$= O\left((\langle\hbar k\rangle_3^r + \langle v\rangle_3^r)\left(1+\left\|\left(\nabla-\frac{v}{2}\right)^{q+r_1+r_3+2}f\right\|\right)\right)$$

This conclude the proof. □

We deduce from the preceding lemma the following bound

**Corollary 3.2.** *There exist two constants $C_0$ and $\hbar_0>0$ such that for any $\hbar<\hbar_0$ and $\|f\|_3 < C_0$, we have $|\varepsilon_\hbar(\Phi,k,v)|\simeq 1$.*

3.2. **The discrete difference norm.** In order to control the linearized collision operator $\mathcal{L}_\hbar$, we need to introduce an adapted norm, which looks like a Sobolev norm with wait.

*Definition* 3.1. We introduce $\mathcal{H}_\hbar$ the Hilbert space of norm $\|\ \|_\hbar$

$$\|g\|_\hbar^2 := \frac{c_d}{\hbar^2}\int\left[(g_1-g_1')^2 M_2 + g_1^2\left(\sqrt{M_2}-\sqrt{M_2'}\right)^2\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1. \tag{3.5}$$

In the same way, we introduce for $\hbar=0$ the space $\mathcal{H}_0$ with norm

$$\|g\|_0^2 := \int\nabla g_1\left(\int B(v_1,v_2)M_2\,\mathrm{d}v_2\right)\nabla g_1\,\mathrm{d}v_1 + \int g_1 v_1\left(\int B(v_1,v_2)M_2\,\mathrm{d}v_2\right)g_1 v_1\,\mathrm{d}v_1 \tag{3.6}$$

$$B(v_1,v_2) := B(M,v_1,v_2) = \int\frac{|\hat{\mathcal{V}}(k)|^2 k\otimes k}{|\varepsilon_0(M,k,v_1)|^2}\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k. \tag{3.7}$$

**Proposition 3.3.** *We fix the dimension $d\geq 2$.*
*There exists $\hbar_0>0$ such that $\hbar\in(0,\hbar_0)$, $\alpha\in(0,1/2]$ and $g$ a test function,*

$$\frac{1}{\hbar^2}\int(g_1-g_1')^2\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-\alpha(v_1\cdot\hat{k})^2}\mathrm{d}k\,\mathrm{d}v_1 + \|\langle v\rangle^{-1/2}g\|^2 \simeq \|g\|_\hbar. \tag{3.8}$$

*where the constant are independent of $\hbar$.*

**Remark 3.1.** *It has been proved in* [DW23] *that*

$$\|g\|_0^2 \simeq \left\|\langle v\rangle^{-\frac{3}{2}}\tfrac{v}{|v|}\cdot\nabla g(v)\right\|^2 + \left\|\langle v\rangle^{-\frac{1}{2}}\left(\mathrm{Id}-\tfrac{v\otimes v}{|v|^2}\right)\nabla g(v)\right\|^2 + \left\|\langle v\rangle^{-\frac{1}{2}}g(v)\right\|^2. \tag{3.9}$$

*Proof.* ***Step* 1.** We want to prove the upper bound

$$\frac{1}{\hbar^2}\int g_1^2\left(M_2{}^\alpha - M_2'{}^\alpha\right)^2\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1 \lesssim \|\langle v\rangle^{-1/2}g\|^2 \tag{3.10}$$

First

$$\int g_1^2\left(M_2{}^\alpha-M_2'{}^\alpha\right)^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2}\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1 = \int g_1{}^2\frac{C\hat{\mathcal{V}}^2(k)}{\hbar^2|k|}\left(e^{-\alpha(v_1\cdot\hat{k}+\hbar k)^2}-e^{-\alpha(v_1\cdot\hat{k})^2}\right)^2\mathrm{d}k\,\mathrm{d}v_1$$

We introduce the change of variable $k\mapsto(|k|,x,\sigma)\in\mathbb{R}^+\times[0,1]\times\mathbb{S}^{d-1}$, defined by

$$k = |k|\left(\sqrt{1-x^2}\sigma + x\tfrac{v_1}{|v_1|}\right),\ |\sigma|=1,\ \sigma\cdot v_1 = 0. \tag{3.11}$$

The Jacobian of this transformation is

$$\mathrm{d}k = |k|^{d-1}\left(1-x^2\right)^{\frac{d-3}{2}}\mathrm{d}|k|\,\mathrm{d}x\,\mathrm{d}\sigma.$$

$$
\begin{aligned}
&\int \frac{\hat{\mathcal{V}}^2(k)}{|k|}\left(e^{-\alpha(v_1\cdot\hat{k}+\hbar k)^2}-e^{-\alpha(v_1\cdot\hat{k})^2}\right)^2 \mathrm{d}k\\
&=\int \frac{\hat{\mathcal{V}}^2(k)}{|k|}\left(\int_0^1 2\alpha\hbar k(|v_1|x+s\hbar|k|)\exp\left(-\alpha(|v_1|x+s\hbar|k|)^2\right)\mathrm{d}s\right)^2 |k|^{d-1}(1-x^2)^{\frac{d-2}{2}}\,\mathrm{d}|k|\,\mathrm{d}x\\
&\le C\hbar^2\alpha^2\int_0^1\int \hat{\mathcal{V}}^2(k)|k|^d\int_{-1}^1 \exp\left(-\alpha(|v_1|x+s\hbar|k|)^2\right)\mathrm{d}x\,\mathrm{d}|k|\,\mathrm{d}s \le \frac{C\hbar^2\alpha^{3/2}}{<v_1>}.
\end{aligned}
$$

***Step* 2.** We treat the lower bound

$$
\frac{1}{\hbar^2}\int g_1^2\left({M_2}^\alpha-{M_2'}^\alpha\right)^2\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2 \gtrsim \||v|\langle v\rangle^{-3/2}g\|^2 - C\hbar^2\|\langle v\rangle^{-1/2}g\|^2 \tag{3.12}
$$

We can restrict our self to the ball $\{k\in\mathbb{R}^d, |k|<1\}$.

$$
\begin{aligned}
\int_{-1}^1\left(\int_0^1(|v_1|x+s\hbar|k|)e^{-\alpha(|v_1|x+s\hbar|k|)^2}\right)^2(1-x^2)^{\frac{d-2}{2}}\,\mathrm{d}x &\gtrsim \int_{-1/2}^{1/2}\left(|v_1x|^2-\frac{\hbar^2|k|}{4}\right)e^{-2\alpha\left(|v_1x|^2+|\hbar k|^2\right)}\mathrm{d}x\\
&\gtrsim \frac{C}{|v_1|}\int_{-2|v_1|}^{2|v_1|}y^2e^{-2\alpha y^2}\,\mathrm{d}y - O\left(\frac{\hbar^2}{\langle v_1\rangle}\right) \ge \frac{C_\alpha|v_1|}{\langle v_1\rangle^3}-O\left(\frac{|k|^2\hbar^2}{\langle v_1\rangle}\right).
\end{aligned}
$$

which conclude the proof.

***Step* 3.** We treat the "differential part" of the norm.

$$
\frac{2}{\hbar^2}\int (g_1-g_1')^2\,\hat{\mathcal{V}}^2(k)\delta_{k\cdot(v_2-v_1-\hbar k)}M_2'\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1 = \frac{C}{\hbar^2}\int (g_1-g_1')^2\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-\frac{(v_1\cdot\hat{k})^2}{2}}\,\mathrm{d}k\,\mathrm{d}v_1.
$$

We begin by cutting the large $k$ and $v_1$.

$$
\begin{aligned}
\int \frac{\hat{\mathcal{V}}^2(k)}{|k|}\mathbb{1}_{|k|\ge K}\,\mathrm{d}k &= O((1+K)^{-s})\\
\int \frac{\hat{\mathcal{V}}^2(k)}{|k|}\exp\left(-\frac{\alpha\,(v_1\cdot k)^2}{|k|^2}\right)\mathrm{d}k &= C\int_{-1}^1\left(1-x^2\right)^{\frac{d-3}{2}}e^{-\alpha|v_1|^2x^2}\,\mathrm{d}x = \frac{C}{\sqrt{\alpha}|v_1|}+O\left(\frac{1}{\sqrt{\alpha}|v_1|}e^{-\frac{\alpha}{2}|v_1|^2}\right)\\
&= \frac{C}{\sqrt{\alpha}|v_1|}+O(\frac{\hbar^2}{|v_1|^2}) \quad \text{if} \quad |v_1|\ge\frac{4}{\alpha}\sqrt{|\log\hbar|}\\
\int \frac{\hat{\mathcal{V}}^2(k)}{|k|}\exp\left(-\frac{\alpha\,(v_1\cdot k)^2}{|k|^2}\right)\mathbb{1}_{|k|\ge K}\,\mathrm{d}k &\le \frac{C}{\langle v_1\rangle}\int_K^\infty \frac{k|^{d-2}}{(1+|k|)^{d+2}}\,\mathrm{d}|k| \le \frac{C}{\langle v_1\rangle K^3}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&\frac{1}{\hbar^{d+1}}\int\left(\mathbb{1}_{|k|\ge\frac{1}{\hbar|\log\hbar|}}+\mathbb{1}_{|v_1|\ge\frac{4}{\alpha}\sqrt{|\log\hbar|}}\right)(g_1-g_1')^2\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-2\alpha\left(v_1\cdot\hat{k}\right)^2}\,\mathrm{d}k\,\mathrm{d}v_1\\
&\qquad\le C\left(\frac{1}{\hbar^{d+1}}\int (g_1-g_1')^2\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-\frac{(v_1\cdot\hat{k})^2}{2}}\,\mathrm{d}k\,\mathrm{d}v_1+\|\langle v\rangle^{-1/2}g\|^2\right).
\end{aligned}
$$

Suppose now that $|v_1|\le C\sqrt{|\log\hbar}$, $|k|\le\frac{1}{\hbar|\log\hbar|}$. We define $k_1,k_2$ by

$$
k_{\frac{1}{2}} := \frac{|k|}{2}\left(\frac{1}{\sqrt{\alpha}}\pm\frac{\sqrt{1-x^2}}{\sqrt{1-\alpha x^2}}\right)\left(\pm\sqrt{1-\alpha x^2}\sigma+\sqrt{\alpha}x\frac{v_1}{|v_1|}\right).
$$

Note that for $\alpha\in(0,1/2)$, the ratio $\frac{|k_1|}{|k|}$ and $\frac{|k_2|}{|k|}$ are inside some segment $[C^{-1},C]$, with $C$ independent of $\hbar$. Hence the Jacobians of $k\mapsto k_1$ and $k\mapsto k_2$ have upper and lower bounds. We deduce that

$$
\begin{aligned}
(g(v_1+\hbar k)-g(v_1))^2e^{-\frac{\alpha(v_1\cdot k)^2}{|k|^2}} &\le 2(g(v_1+\hbar k_1)-g(v_1))^2e^{-\frac{(v_1\cdot\hat{k}_1)^2}{2}}\\
&\quad+2C(g(v_1+\hbar k)-g(v_1+\hbar k_1))^2e^{-\frac{((v_1+\hbar k_1)\cdot\hat{k}_2)^2}{2}}.
\end{aligned}
$$

Using Assumption 2, we deduce that

$$\int (g_1-g_1')^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{\alpha(v_1\cdot k)^2}{|k|^2}} \mathbb{1}_{\substack{|k|\le \frac{1}{\hbar|\log\hbar|}\\ |v_1|\lesssim\sqrt{|\log\hbar|}}} \,\mathrm{d}k\,\mathrm{d}v_1 \lesssim \int (g(v_1)-g(v_1+\hbar k_1))^2 \frac{\hat{\mathcal{V}}^2(k_1)}{|k_1|} e^{-\frac{(v_1\cdot\hat{k_1})^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1$$
$$+\int \big(g(v_1+\hbar k)-g(v_1-\hbar k_1)\big)^2 \frac{\hat{\mathcal{V}}^2(k_2)}{|k_2|} e^{-\frac{((v_1+\hbar k_1)\cdot\hat{k}_2)^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1$$
$$\lesssim \int (g_1-g_1')^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} \varepsilon^{-\frac{(v_1\cdot\hat{k})^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1$$

***Step* 4.** We prove the integrability near 0:

$$\int_{|v_1|<1} g(v_1)^2\,\mathrm{d}v_1 \le C\int_{\substack{2<|k|<3\\|v_1|<1}} (g(v_1+k)-g(v_1))^2\,\mathrm{d}v_1 + \int_{\substack{2<|k|<3\\|v_1|<1}} (g(v_1+k))^2\,\mathrm{d}v_1$$

Using the convex inequality, one can bound $\int_{|v_1|<1} g(v_1)^2\,\mathrm{d}v_1$ by

$$C\left\lfloor\hbar^{-1}\right\rfloor \sum_{n=0}^{\lfloor\hbar^{-1}\rfloor-1} \int_{\substack{2<|k|<3\\|v_1|<1}} \left(g\left(v_1+\frac{(n+1)}{\lfloor\hbar^{-1}\rfloor}k\right) - g\left(v_1+\frac{n}{\lfloor\hbar^{-1}\rfloor}k\right)\right)^2 \mathrm{d}v_1\,\mathrm{d}k + C\||v|\langle v\rangle^{-3/2}g\|^2$$
$$\le C\int_{\substack{2<|k|<3\\|v_1|<1+3\hbar}} \frac{\left(g\left(v_1+\frac{k}{\lfloor\hbar^{-1}\rfloor}\right)-g(v_1)\right)^2}{\lfloor\hbar^{-1}\rfloor^{-2}} \mathrm{d}v_1\,\mathrm{d}k + C\||v|\langle v\rangle^{-3/2}g\|^2$$
$$\le \int (g_1-g_1')^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{1}{2}(v_1\cdot\hat{k})^2} \,\mathrm{d}k\,\mathrm{d}v_1 + C\||v|\langle v\rangle^{-3/2}g\|^2,$$

where we use the change of variable $k\mapsto \frac{\hbar^{-1}}{\lfloor\hbar^{-1}\rfloor}k$ and that $2<|k|<3$, $|v_1|<4$.

Hence

$$\|\langle v\rangle^{-1/2}g\|^2 \lesssim \int \frac{(g_1-g_1')^2}{\hbar^2} \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{1}{2}(v_1\cdot\hat{k})^2} k\,\mathrm{d}v_1 + C\||v|\langle v\rangle^{-3/2}g\|^2$$

***Step* 5.** Conclusion

Using Step 1,

$$\|g\|_\hbar^2 \lesssim \frac{1}{\hbar^2}\int (g_1-g_1')^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-2\alpha(v_1\cdot\hat{k})^2} \,\mathrm{d}k\,\mathrm{d}v_1 + \|\langle v\rangle^{-1/2}g\|^2.$$

Using Steps 2, 3 and 4,

$$\frac{1}{\hbar^2}\int (g_1-g_1')^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-2\alpha(v_1\cdot\hat{k})^2} \,\mathrm{d}k\,\mathrm{d}v_1 + \|\langle v\rangle^{-1/2}g\|^2 \lesssim \|g\|_\hbar - C\hbar^2\|\langle v\rangle^{-1/2}g\|^2.$$

□

**Proposition 3.4.** *There exists a constant $C$ such that for any test function $g$,*

$$0\le\hbar<\hbar' \Rightarrow \|g\|_{\hbar'} \le C\|g\|_\hbar. \tag{3.13}$$

*Proof.* We recall that

$$\|g\|_\hbar^2 \simeq \frac{1}{\hbar^2}\int (g(v_1)-g(v_1+\hbar k))^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{(v_1\cdot\hat{k})^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1 + \|\langle v\rangle^{-1/2}g\|^2$$

Hence we only need to focus on the first terms.

$$\frac{1}{\hbar^2}\int (g(v_1)-g(v_1+\hbar k))^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{(v_1\cdot\hat{k})^2}{2}} k\,\mathrm{d}v_1$$
$$\le \frac{2}{\hbar^2}\int \left[\left(g(v_1)-g(v_1+\frac{\hbar}{2}k)\right)^2 + \left(g(v_1+\frac{\hbar}{2}k)-g(v_1+\hbar k)\right)^2\right] \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{(v_1\cdot\hat{k})^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1$$
$$\le \frac{1}{(\hbar/2)^2}\int \left(g(v_1)-g(v_1+\frac{\hbar}{2}k)\right)^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{(v_1\cdot\hat{k})^2}{2}} \,\mathrm{d}k\,\mathrm{d}v_1$$

For $q\in]\frac{1}{2},1]$,

$$\frac{1}{\hbar^2}\int (g(v_1)-g(v_1+\hbar k))^2\,\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-(v_1\cdot\hat{k})^2}\,\mathrm{d}k\,\mathrm{d}v_1 = \frac{q^2}{(q\hbar)^2}\int (g(v_1)-g(v_1+q\hbar k))^2\,\frac{\hat{\mathcal{V}}^2(qk)}{|qk|}e^{-(v_1\cdot\hat{k})^2}\,\frac{\mathrm{d}k}{q^d}\,\mathrm{d}v_1$$
$$\lesssim \frac{1}{\hbar^2}\int (g(v_1)-g(v_1+q\hbar k))^2\,\frac{\hat{\mathcal{V}}^2(k)}{|k|}e^{-(v_1\cdot\hat{k})^2}\,\mathrm{d}k\,\mathrm{d}v_1$$

Finally we use that for any $0<\hbar<\hbar'$, there exists a couple $(r,q)\in\mathbb{N}\times]1/2,1]$ such that $\hbar=q2^{-r}\hbar'$. □

**3.3. Decomposition of the quadratic form $\int g\mathcal{L}_\hbar g$.** We can decompose $\int g\mathcal{L}_\hbar g$ into three peaces:

$$\int g\mathcal{L}_\hbar g = \frac{c_d}{4\hbar^2}\int\left(g_1\sqrt{M_2}+g_2\sqrt{M_1}-g_1'\sqrt{M_2'}-g_2'\sqrt{M_1'}\right)^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$

$$=\frac{c_d}{2\hbar^2}\int\left[(g_1-g_1')^2M_2+g_1'^2\left(\sqrt{M_2}-\sqrt{M_2'}\right)^2\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 \tag{3.14}$$

$$+\frac{c_d}{\hbar^2}\int\left[(g_1-g_1')g_1'\sqrt{M_2}\left(\sqrt{M_2}-\sqrt{M_2'}\right)\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 \tag{3.15}$$

$$+\frac{c_d}{2\hbar^2}\int\left(g_1\sqrt{M_2}-g_1'\sqrt{M_2'}\right)\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 \tag{3.16}$$

One can identify (3.14) with $\|g\|_\hbar^2/2$. The goal of this section is to study the remaining parts.

This decomposition will be necessary in the proof of Proposition 3.7, which is the crucial step of the proof.

We need to introduce a cutoff function.

*Definition* 3.2. Fix $\varphi:\mathbb{R}_+\to[0,1]$ an increasing function, with $\varphi(1)=1$, $\varphi(0)=0$ and $\varphi'$ is supported in $[1/3,2/3]$. We define $\varphi_K(v):=\varphi(K|v|)$.

**Proposition 3.5.** *We fix the dimension $d\geq 2$.*

*There exists a kernel $\mathcal{K}_\hbar(v_1,v_2)$ such that*

$$\frac{c_d}{\hbar^2}\int\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\sqrt{M_2}\,\mathrm{d}k\,\mathrm{d}v_2=\int\mathcal{K}_\hbar(v_1,v_2)g(v_2)\,\mathrm{d}v_2 \tag{3.17}$$

*with the bound*

$$|\mathcal{K}_\hbar(v_1,v_2)|\leq\frac{C(1+|v_1|^2+|v_2|^2)\sqrt[5]{M_1M_2}}{|v_1-v_2|^3}. \tag{3.18}$$

*In addition, for almost all $(v_1,v_2)$,*

$$\lim_{\hbar\to0}\mathcal{K}_\hbar(v_1,v_2)=\mathcal{K}_0(v_1,v_2):=-\left(\nabla_1-\frac{v_1}{2}\right)\otimes\left(\nabla_2-\frac{v_2}{2}\right):\left(B(v_1,v_2)\sqrt{M_1M_2}\right). \tag{3.19}$$

*Finally, one can cutoff the singularity at $\{v_1=v_2\}$:*

$$\frac{c_d}{\hbar^2}\int\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\left(h_1\sqrt{M_2}-h_1'\sqrt{M_2'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\sqrt{M_2}\,\mathrm{d}k\,\mathrm{d}v_2 \tag{3.20}$$
$$=\int\left(\varphi_K(v_1-v_2)\mathcal{K}_\hbar(v_1,v_2)-\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)g(v_2)\,\mathrm{d}v_2+O\left(\left(\tfrac{1}{K}+\hbar K^4\right)\|g\|_\hbar\|h\|_\hbar\right).$$

*where we denote*

$$\rho_K(v_1-v_2,\frac{v_1+v_2}{2}):=\frac{K\varphi'(K|v_1-v_2|)}{|v_1-v_2|^2}\int_{<v_1-v_2>^\perp}\frac{|k|^2|\hat{\mathcal{V}}|^2(k)\,\mathrm{d}k}{|\varepsilon_0(M,k,\frac{v_1+v_2}{2})|^2}. \tag{3.21}$$

*Proof.* For $h\in L^2(\frac{\mathrm{d}v_1}{\langle v_1\rangle})$ and $g\in L^2(\frac{\mathrm{d}v_1}{\langle v_1\rangle})$,

$$\frac{2c_d}{\hbar^2}\int h_1\sqrt{M_2}\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2$$
$$=\frac{c_d}{\hbar^2}\int\left(h_1\sqrt{M_2}-h_1'\sqrt{M_2'}\right)\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$
$$=\int h_1g_2\mathcal{K}_\hbar(v_1,v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2$$

where, denoting $v_1'' := v_1 - \hbar k$ and $v_2'' := v_2 + \hbar k$, we define

$$\mathcal{K}_\hbar(v_1,v_2) := \int \frac{c_d|\hat{\mathcal{V}}(k)|^2}{\hbar^2}\left(\frac{\sqrt{M_1M_2}\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2} - \frac{\sqrt{M_1''M_2}\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1'')|^2} - \frac{\sqrt{M_1M_2''}\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1)|^2} + \frac{\sqrt{M_1M_2}\delta_{k\cdot(v_2-v_1+\hbar k)}}{|\varepsilon_\hbar(M,k,v_1'')|^2}\right)\mathrm{d}k,$$

Using that $|\hat{\mathcal{V}}(k)| \le C(1+|k|)^{d+1}$, Proposition 3.1 and that if $k\cdot(v_1-v_2)=0$,

$$|v_1-\hbar k|^2 + |v_2|^2 = \frac{|v_1-v_2|^2}{2} + \frac{|v_1+v_2|^2}{2} + (v_1+v_2)\cdot k + |k|^2 \ge \frac{|v_1-v_2|^2}{2} + \frac{|v_1+v_2|^2}{4} \ge \frac{|v_1|^2+|v_2|^2}{2},$$

one can prove the trivial bound

$$|\mathcal{K}_\hbar(v_1,v_2)| \lesssim \frac{\sqrt[4]{M_1M_2}}{\hbar^2|v_1-v_2|}.$$

***Step* 1.** We split $\mathcal{K}_\hbar(v_1,v_2)$ into two pieces :

$$\mathcal{K}_\hbar =: \mathcal{K}_\hbar^< + \mathcal{K}_\hbar^>, \tag{3.22}$$

$$\mathcal{K}_\hbar^>(v_1,v_2) := \int_{|k|\ge\frac{|v_1-v_2|}{4\hbar}} \frac{c_d|\hat{\mathcal{V}}(k)|^2}{\hbar^2}\left(\frac{\sqrt{M_1M_2}\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2} - \frac{\sqrt{M_1''M_2}\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1'')|^2} - \frac{\sqrt{M_1M_2''}\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1)|^2} + \frac{\sqrt{M_1M_2}\delta_{k\cdot(v_2-v_1+\hbar k)}}{|\varepsilon_\hbar(M,k,v_1'')|^2}\right)\mathrm{d}k \tag{3.23}$$

Using that $|\hat{\mathcal{V}}(k)|^2 \le C(1+|k|)^{d+1}$ and Proposition 3.1,

$$|\mathcal{K}_\hbar^>(v_1,v_2)| \lesssim \frac{\sqrt[4]{M_1M_2}}{\hbar^2|v_1-v_2|}\left(\left(1+\frac{|v_1-v_2|}{\hbar}\right)^{-2} + \left(1+\frac{|v_1-v_2|}{\hbar}\right)^{-d-1}\frac{|v_1-v_2|^{d-1}}{\hbar^{d-1}}\right) \lesssim \frac{\sqrt[4]{M_1M_2}}{|v_1-v_2|^3}.$$

In addition, we have that $\mathcal{K}_\hbar^>(v_1,v_2) \to 0$ almost every where

***Step* 2.** For $k \in < v_1-v_2 >^\perp$ with $|k| \le \frac{|v_1-v_2|}{2\hbar}$, we introduce

$$\delta k := |\delta k|\frac{v_1-v_2}{|v_1-v_2|},\ |\delta k| = \frac{|v_2-v_1|}{2\hbar}\left(1-\sqrt{1-\frac{4\hbar^2|k|^2}{|v_2-v_1|^2}}\right) = \frac{\hbar|k|^2}{|v_1-v_2|} + o\left(\frac{\hbar|k|^2}{|v_1-v_2|}\right).$$

Note that $k+\delta k$ lays in the sphere $\{k', k'\cdot(v_2-v_1-\hbar k')=0\}$. The Jacobian of the transformation $k \mapsto k' := k+\delta k$ is

$$\delta_{k'\cdot(v_2-v_1-\hbar k')}\,\mathrm{d}k' = \Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right)\delta_{k\cdot(v_2-v_1)}\,\mathrm{d}k$$

$$\Lambda(x) := \frac{1}{\sqrt{1-x^2}},\ \forall x\in[0,1/4],\ \Lambda(x) = 1+\frac{x^2}{2}+o(x^2).$$

We want to bound

$$\frac{\sqrt{M_1M_2}|\hat{\mathcal{V}}(k+\delta k)|^2\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right)}{|\varepsilon_\hbar(M,k+\delta k,v_1)|^2} - \frac{\sqrt{M_1''M_2}|\hat{\mathcal{V}}(k)|^2}{|\varepsilon_\hbar(M,k,v_1'')|^2} - \frac{\sqrt{M_1M_2''}|\hat{\mathcal{V}}(k)|^2}{|\varepsilon_\hbar(M,k,v_1)|^2} + \frac{\sqrt{M_1M_2}|\hat{\mathcal{V}}(k-\delta k)|^2\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right)}{|\varepsilon_\hbar(M,k-\delta k,v_1''+\hbar\delta k)|^2}$$

Note that as $k\cdot\delta k = 0$, one have

$$|\varepsilon_\hbar(M,k-\delta k,v_1''+\hbar\delta k)|^2 = |\varepsilon_\hbar(M,k-\delta k,v_1'')|^2.$$

Using Proposition 3.1 and $|\hat{\mathcal{V}}(k)|^2 + |\hat{\mathcal{V}}'(k)|^2 + |\hat{\mathcal{V}}''(k)|^2 = O(\langle k\rangle^{-d-3})$, one can bound the derivative of $\varpi(k,v_1) := \frac{|\hat{\mathcal{V}}|^2(k)}{|\varepsilon_\hbar(M,k,v_1)|^2}$,

$$|\varpi(k,v_1)| + |\nabla\varpi(k,v_1)| + |\nabla^2\varpi(k,v_1)| \lesssim \frac{\langle k\rangle^2\langle v_1\rangle^2}{|k|^2(|k|+1)^{d+5}}.$$

Hence using the Taylor formula and that $k \cdot \delta k = 0$,

$$\varpi(k+\delta k, v_1)\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right) - \varpi(k, v_1-\hbar k) - \varpi(k,v_1) + \varpi(k-\delta k, v_1-\hbar k)\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right)$$
$$= (\delta k - \delta k)\partial_k \varpi(k,v_1) + (\hbar k - \hbar k)\partial_{v_1}\varpi(k,v_1) + O\left(\frac{\langle k\rangle^2\langle v_1\rangle^2\left(|\delta k|^2 + \hbar\,|k|\,|\,|\delta k| + \hbar^2\,|k|^2\right)}{|k|^2(1+|k|)^{d+5}}\right).$$

Using that $k\cdot(v_2-v_1)=0$,

$$\frac{\left(\sqrt{M_1''}-\sqrt{M_1}\right)\sqrt{M_2}}{|\varepsilon_\hbar(M,k,v_1'')|^2} + \frac{\sqrt{M_1}\left(\sqrt{M_2''}-\sqrt{M_2}\right)}{|\varepsilon_\hbar(M,k,v_1)|^2} = \frac{\hbar k\cdot\frac{v_1-v_2}{2}\sqrt{M_1M_2}}{|\varepsilon_\hbar(M,k,v_1)|^2} + O\left(\hbar^2|k|^2\sqrt[5]{M_1M_2}\right)$$
$$= O\left(\hbar^2|k|^2\sqrt[5]{M_1M_2}\right)$$

Integrating with respect to $k$, this gives the bound

$$\left|\mathcal{K}_\hbar^{2,<}(v_1,v_2)\right| \le \frac{C\sqrt[5]{M_1M_2}}{|v_1-v_2|^3}. \tag{3.24}$$

***Step* 3.** It sufficient to prove for almost all $(v_1,v_2,k)$ the convergence of

$$\frac{\sqrt{M_1M_2}\hat{\mathcal{V}}^2(k+\delta k)}{|\varepsilon_\hbar(M,k+\delta k,v_1)|^2}\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right) - \frac{\sqrt{M_1''M_2}\hat{\mathcal{V}}^2(k)}{|\varepsilon_\hbar(M,k,v_1'')|^2} - \frac{\sqrt{M_1M_2''}\hat{\mathcal{V}}^2(k)}{|\varepsilon_\hbar(M,k,v_1)|^2} + \frac{\sqrt{M_1M_2}\hat{\mathcal{V}}(k-\delta k)}{|\varepsilon_\hbar(M,k-\delta k,v_1'')|^2}\Lambda\left(\frac{2\hbar|k|}{|v_1-v_2|}\right)$$

when $\hbar\to 0$. Then we can apply the previous estimation to apply the dominated convergence Theorem. The convergence comes from a limited development, and the computation are similar than the one for *a priori* estimations.

In order to identify the limit, we proceed by duality. Let $g, f$ two smooth, compactly supported functions.

$$\int h_1g_2\mathcal{K}_\hbar(v_1,v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= \frac{c_d}{\hbar^2}\int\left(h_1\sqrt{M_2}-h_1'\sqrt{M_2'}\right)\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$
$$= \frac{c_d}{\hbar^2}\int\left(h_1''\sqrt{M_2}-h_1\sqrt{M_2'}\right)\left(g_2\sqrt{M_1''}-g_2'\sqrt{M_1}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1'')|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$
$$= -c_d\int k\cdot\left(\nabla h_1+\frac{v_2}{2}h_1\right)\left(\nabla g_2+\frac{v_1}{2}g_2\right)\cdot k\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\sqrt{M_1M_2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1 + O(\hbar)$$
$$= -\int h_1g_2\left[\left(\nabla_1-\frac{v_1}{2}\right)\otimes\left(\nabla_2-\frac{v_2}{2}\right) : \int\frac{k\otimes k\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\sqrt{M_1M_2}\,\mathrm{d}k\right]\mathrm{d}v_2\,\mathrm{d}v_1 + O(\hbar)$$

***Step* 4.** We treat now the case of dimension 3. Using the same strategy than for $\mathcal{K}_\hbar(v_1,v_2)$,

$$\frac{c_d}{\hbar^2}\int\left(h_1\sqrt{M_2}-h_1'\sqrt{M_2'}\right)\left(g_2\sqrt{M_1}-g_2'\sqrt{M_1'}\right)\varphi_K(v_1-v_2)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$
$$= \int h_1g_2\left(\varphi_K(v_1-v_2)\mathcal{K}_\hbar(v_1,v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 + \mathcal{K}_\hbar^K(v_1,v_2)\right)\mathrm{d}v_1\,\mathrm{d}v_2$$

with

$$\mathcal{K}_\hbar^K(v_1,v_2) = \int\frac{c_d|\hat{\mathcal{V}}(k)|^2}{\hbar^2}\left(\frac{\sqrt{M_1M_2}}{|\varepsilon_\hbar(M,k,v_1'')|^2}[\varphi_K(v_1-v_2-2\hbar k)-\varphi_K(v_1-v_2)]\delta_{k\cdot(v_2-v_1+\hbar k)}\right.$$
$$\left. - \left(\frac{\sqrt{M_1''M_2}}{|\varepsilon_\hbar(M,k,v_1'')|^2}+\frac{\sqrt{M_1M_2''}}{|\varepsilon_\hbar(M,k,v_1)|^2}\right)[\varphi_K(v_1-v_2-\hbar k)-\varphi_K(v_1-v_2)]\delta_{k\cdot(v_2-v_1)}\right)\mathrm{d}k,$$

We recall that $\nabla^k\varphi_K = O(K^k)$ and

$$\nabla\varphi_K(v) = K\varphi'(K|v|)\frac{v}{|v|},\ \nabla^2\varphi_K(v) = K^2\varphi''(K|v|)\frac{v^{\otimes 2}}{|v|^2} + K\varphi'(K|v|)\left(\frac{\mathrm{Id}}{|v|}-\frac{v^{\otimes 2}}{|v|^3}\right).$$

• If $k\cdot(v_2-v_1+\hbar k)=0$, we have $|\hbar k|\le|v_1-v_2|$ and

$$\begin{aligned}
&\varphi_K(v_1-v_2-2\hbar k)-\varphi_K(v_1-v_2)\\
&=2\left(-\hbar k\cdot\nabla\varphi_K(v_1-v_2)+\hbar^2k^{\otimes 2}:\nabla^2\varphi_K(v_1-v_2)\right)+O\left((\hbar K|k|)^3\right)\\
&=\frac{2K\varphi'(K|v_1-v_2|)}{|v_1-v_2|}\left(-(v_1-v_2)\cdot(\hbar k)+|\hbar k|^2-\frac{(\hbar k\cdot(v_1-v_2))^2}{|v_1-v_2|^2}\right)-\frac{2K^2\varphi''(K|v_1-v_2|)(\hbar k\cdot(v_1-v_2))^2}{|v_1-v_2|^2}\\
&\qquad+O\left((\hbar K|k|)^3\right)\\
&=-\frac{2K\varphi'(K|v_1-v_2|)\hbar^4|k|^4}{|v_1-v_2|^3}-\frac{2K^2\varphi''(K|v_1-v_2|)\hbar^4|k|^4}{|v_1-v_2|^2}+O\left((\hbar K|k|)^3\right)=O\left((\hbar K|k|)^3\right)
\end{aligned}$$

• If $k\cdot(v_2-v_1)=0$, we have

$$\begin{aligned}
\varphi_K(v_1-v_2-\hbar k)-\varphi_K(v_1-v_2)&=\left(-\hbar k\cdot\nabla\varphi_K(v_1-v_2)+\frac{\hbar^2k^{\otimes 2}}{2}:\nabla^2\varphi_K(v_1-v_2)\right)+O\left((\hbar K|k|)^3\right)\\
&=\frac{K\varphi'(K|v_1-v_2|)\hbar^2|k|^2}{2|v_1-v_2|}+O\left((\hbar K|k|)^3\right)
\end{aligned}$$

Integrating with respect to $k$, and using Proposition 3.1,

$$\begin{aligned}
\mathcal{K}_\hbar^K(v_1,v_2)&=-\rho_K(v_1-v_2,\frac{v_1+v_2}{2})+\int O\left(\hbar K^3|\hat{\mathcal{V}}(k)|^2|k|^3\right)\left(\delta_{k\cdot(v_2-v_1+\hbar k)}+\delta_{k\cdot(v_2-v_1)}\right)\mathrm{d}k\\
&=-\rho_K(v_1-v_2,\frac{v_1+v_2}{2})+O\left(\tfrac{\hbar K^3\sqrt{M_1M_2}}{|v_1-v_2|}\right)
\end{aligned}\tag{3.25}$$

We use that

$$k\cdot(v_1-v_2)=0\Rightarrow|\varepsilon_0(M,k,v_1)|^2=\left|1+\hat{\mathcal{V}}(k)\int\frac{k\cdot\nabla M(v_*)}{k\cdot(v_1-v_*)-i0}\right|^2=\left|\varepsilon_0\left(M,k,\tfrac{v_1+v_2}{2}\right)\right|^2.$$

This leads to the following estimation:

$$\begin{aligned}
&\int\int h(v_1)g(v_2)\left(\mathcal{K}_\hbar^K(v_1,v_2)-\varphi_K(v_1-v_2)\mathcal{K}_\hbar(v_1,v_2)+\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)\mathrm{d}v_1\,\mathrm{d}v_2\\
&\qquad=\int h_1g_1M_1\int\frac{2\pi|k|^2|\hat{\mathcal{V}}|^2(k)\,\mathrm{d}k}{|\varepsilon_0(M,k,v_1)|^2}\mathrm{d}v_1+O\left(\left(\tfrac{1}{K}+\hbar K^4\right)\|h\|_\hbar\|g\|_\hbar\right).
\end{aligned}\tag{3.26}$$

***Step* 5.**

$$\begin{aligned}
&\left|\frac{1}{\hbar^2}\int\left(g_1\sqrt{M_2}-g_1'\sqrt{M_2'}\right)\left(g_{n,2}\sqrt{M_1}-g_2'\sqrt{M_2'}\right)(1-\varphi_K(v_1-v_2))\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,v_1,k)|^2}\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2\right|\\
&\qquad\lesssim\frac{1}{\hbar^2}\int_{|v_1-v_2|\le\frac{1}{K}}\left(g_1\sqrt{M_2}-g_1'\sqrt{M_2'}\right)^2\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2\\
&\qquad\lesssim\frac{1}{\hbar^2K}\int\left(g_1e^{-\frac{(\hat{k}\cdot v_1)^2}{2}}-g_1'e^{-\frac{(\hat{k}\cdot v_1-\hbar k)^2}{2}}\right)^2\frac{\hat{\mathcal{V}}(k)^2}{|k|}\mathrm{d}k\,\mathrm{d}v_1=O\left(\frac{\|g\|_\hbar^2}{K}\right).
\end{aligned}$$

This conclude the proof. □

**Proposition 3.6.** *We fix the dimension $d\ge2$. There exists a kernel $\tilde{\mathcal{K}}_\hbar(v_1,v_2)$ such that*

$$\frac{c_d}{\hbar^2}\int(g_1-g_1')g_1\sqrt{M_2}\left(\sqrt{M_2}-\sqrt{M_2'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1=\int\tilde{\mathcal{K}}_0(v_1)g^2(v_1)\,\mathrm{d}v_1+O\left(\hbar\|g\|_\hbar^2\right)\tag{3.27}$$

*where*

$$\tilde{\mathcal{K}}_0(v_1):=\frac{c_d}{4}\nabla_1\cdot\left(v_1\int k\otimes k\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\mathrm{d}kM_2\,\mathrm{d}v_2\right)=O\left(\frac{1}{\langle v_1\rangle^3}\right).\tag{3.28}$$

*Proof.*

$$\begin{aligned}
&\frac{c_d}{\hbar^2}\int(g_1-g_1')g_1\sqrt{M_2}\left(\sqrt{M_2}-\sqrt{M_2'}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
&=\frac{c_d}{4\hbar^2}\int\left[\left(g_1^2-g_1'^2\right)(M_2-M_2')+(g_1-g_1')^2\left(\sqrt{M_2}-\sqrt{M_2'}\right)^2\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1
\end{aligned}\tag{3.29}$$

Using the change of variable $v_1 \mapsto v_1 - \hbar k$,

$$(3.29) = \int g_1^2 \tilde{\mathcal{K}}_\hbar(v_1)\,\mathrm{d}v_1 + \frac{c_d}{\hbar^2}\int (g_1-g_1')^2\left(\sqrt{M_2}-\sqrt{M_2'}\right)^2 \frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$

$$\tilde{\mathcal{K}}_\hbar(v_1) := \frac{c_d}{4\sqrt{2\pi}\hbar^2}\int \frac{\hat{\mathcal{V}}(k)^2}{|k|}\left(\frac{e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}} - e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}}}{|\varepsilon_\hbar(M,k,v_1)|^2} - \frac{e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}} - e^{-\frac{(v_1\cdot\hat{k}+2\hbar|k|)^2}{2}}}{|\varepsilon_\hbar(M,k,v_1-\hbar k)|^2}\right)\mathrm{d}k$$

***Step* 1.** We denote

$$(3.30)\qquad \varpi(k, v_1\cdot\hat{k},\hbar) := \frac{1}{\hbar^2|k|^2}\left(\frac{e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}} - e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}}}{|\varepsilon_\hbar(M,k,v_1)|^2} - \frac{e^{-\frac{(v_1\cdot\hat{k}+\hbar|k|)^2}{2}} - e^{-\frac{(v_1\cdot\hat{k}+2\hbar|k|)^2}{2}}}{|\varepsilon_\hbar(M,k,v_1-\hbar k)|^2}\right).$$

Using a Taylor formula, and introducing $w := v_1\cdot\hat{k}$,

$$\varpi(k,w,\hbar) = \int_{[0,1]^2}\frac{(w+(s+s')\hbar|k|)^2-1}{|\varepsilon_\hbar(M,k,w\hat{k})|^2}e^{-\frac{((w+(s+s')\hbar|k|)}{2}}\,\mathrm{d}s\,\mathrm{d}s'$$
$$+\int_{[0,1]^2}\frac{(w+s\hbar|k|)}{|\varepsilon_\hbar(M,k,(w+s'\hbar|k|)\hat{k})|^4}\hat{k}\cdot\nabla_v|\varepsilon_\hbar(M,k,(w+s'\hbar|k|)\hat{k})|^2 e^{-\frac{(w+s\hbar|k|)^2}{2}}\,\mathrm{d}s\,\mathrm{d}s'.$$

One have $\forall (k,w,\hbar)\in\mathbb{R}^d\times\mathbb{R}\times(0,\infty)$,

$$(3.31)\qquad |\varpi(k,w,\hbar)| \lesssim \int_0^2 e^{-\frac{(w+s\hbar|k|)^2}{4}}\,\mathrm{d}s$$

$$(3.32)\qquad \varpi(k,w,0) = \left(\frac{w^2-1}{|\varepsilon_0(M,k,w\hat{k})|^2} + \frac{w\hat{k}\cdot\nabla_v|\varepsilon_0(M,k,w\hat{k})|^2}{|\varepsilon_0(M,k,w\hat{k})|^4}\right)e^{-\frac{w^2}{2}} = \frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{-w}{|\varepsilon_0(M,k,w\hat{k})|^2}e^{-\frac{w^2}{2}}\right)$$

$$(3.33)\qquad |\varpi(k,w,\hbar)-\varpi(k,w,0)| \lesssim \hbar(1+|k|)\int_0^2 e^{-\frac{(w+s\hbar|k|)^2}{4}}\,\mathrm{d}s \text{ for } \hbar|k|\le 1$$

This last estimate can be deduce from 2.9 and using the same method than before.

Using the change of variable $k\mapsto(\sigma,x,|k|)$ introduce in (3.11), we have

$$(3.34)\qquad \tilde{\mathcal{K}}_\hbar(v_1) = \frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int \hat{\mathcal{V}}^2(k)|k|^d\left(1-x^2\right)^{\frac{d-3}{2}}\varpi(k,|v_1|x,\hbar)\,\mathrm{d}x\,\mathrm{d}|k|$$

We introduce

$$(3.35)\qquad \tilde{\mathcal{K}}_0(v_1) := \frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int \hat{\mathcal{V}}^2(k)|k|^d\left(1-x^2\right)^{\frac{d-3}{2}}\varpi(k,|v_1|x,0)\,\mathrm{d}x\,\mathrm{d}|k|.$$

We split the difference $\tilde{\mathcal{K}}_\hbar(v_1)-\tilde{\mathcal{K}}_0(v_1)$ into two part

$$(3.36)\qquad \frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int_{|k|>|\hbar|^{-1}}\hat{\mathcal{V}}^2(k)|k|^d\left(1-x^2\right)^{\frac{d-3}{2}}\left(\varpi(k,|v_1|x,\hbar)-\varpi(k,|v_1|x,0)\right)\mathrm{d}x\,\mathrm{d}|k|$$

$$(3.37)\qquad +\frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int_{|k|<|\hbar|^{-1}}\hat{\mathcal{V}}^2(k)|k|^d\left(1-x^2\right)^{\frac{d-3}{2}}\left(\varpi(k,|v_1|x,\hbar)-\varpi(k,|v_1|x,0)\right)\mathrm{d}x\,\mathrm{d}|k|$$

We get the following bound: using (3.31),

$$(3.36) \lesssim \int_0^2\int_{|k|>|\hbar|^{-1}}\hat{\mathcal{V}}^2(k)|k|^d e^{-\frac{(x|v_1|+s\hbar|k|)^2}{4}}\,\mathrm{d}x\,\mathrm{d}|k|\,\mathrm{d}s$$
$$\lesssim \int_{|k|>|\hbar|^{-1}}\hat{\mathcal{V}}^2(k)|k|^d\,\mathrm{d}|k|\int_{\mathbb{R}}e^{-\frac{(x|v_1|)^2}{2}}\,\mathrm{d}x \lesssim \frac{\hbar}{<v_1>},$$

and using (3.33)

$$(3.37) \lesssim \hbar\int_0^2\int \hat{\mathcal{V}}^2(k)|k|^d(1+|k|)e^{-\frac{(x|v_1|+s\hbar|k|)^2}{4}}\,\mathrm{d}x\,\mathrm{d}|k|\,\mathrm{d}s \lesssim \frac{\hbar}{<v_1>}.$$

The second line is obtained using (3.33). This gives the approximation results

$$(3.38)\qquad \mathcal{K}_\hbar(v_1) = \mathcal{K}_0(v_1) + O\left(\frac{\hbar}{\langle v_1\rangle}\right).$$

In order to bound $\tilde{\mathcal{K}}_0(v_1)$, we split it into two parts:

$$\tilde{\mathcal{K}}_0(v_1) = \frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int_{|k|<|\hbar|^{-1}}\left(\left(1-x^2\right)^{\frac{d-3}{2}}-1\right)\hat{\mathcal{V}}^2(k)|k|^d\varpi(k,|v_1|x,0)\,\mathrm{d}x\,\mathrm{d}|k| \tag{3.39}$$

$$+\frac{c_d|\mathbb{S}^{d-2}|}{4\sqrt{2\pi}}\int_{|k|<|\hbar|^{-1}}\hat{\mathcal{V}}^2(k)|k|^d\varpi(k,|v_1|x,0)\,\mathrm{d}x\,\mathrm{d}|k|. \tag{3.40}$$

Using that for $|x|\leq\frac{1}{2}$, $(1-(1-x^2)^{(d-3)/2})\lesssim x^2$, and Inequality (3.31),

$$(3.39)\lesssim\int_{|x|\leq\frac{1}{2}}|\hat{\mathcal{V}}(k)|^2|k|^d x^2 e^{-\frac{(x|v_1|)^2}{4}}\,\mathrm{d}x\,\mathrm{d}|k|+\int_{|x|\leq\frac{1}{2}}|\hat{\mathcal{V}}(k)|^2|k|^d\left|\left(1-x^2\right)^{\frac{d-3}{2}}-1\right|e^{-\frac{|v_1|^2}{8}}\,\mathrm{d}x\,\mathrm{d}|k| \lesssim\frac{1}{<v_1>^3}.$$

Using the equality (3.32),

$$(3.40)=\frac{c_d|\mathbb{S}^{d-2}|}{\sqrt{2\pi}}\int_{|k|<1/\hbar}|\hat{\mathcal{V}}(k)|^2|k|^d\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{-w}{|\varepsilon_0(M,k,w\hat{k})|^2}e^{-\frac{w^2}{2}}\right)\Bigg|_{w=v_1\cdot\hat{k}}\mathrm{d}x\,\mathrm{d}k=O\left(\frac{e^{-\frac{|v_1|^2}{4}}}{\langle v_1\rangle}\right).$$

We deduce that $\tilde{\mathcal{K}}_0(v_1)=O(\frac{1}{\langle v_1\rangle^3})$.

***Step* 2.** We treat now the remaining part

$$\begin{aligned}
&\int(g_1-g_1')^2\left(\sqrt{M_2}-\sqrt{M_2'}\right)^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
\lesssim&\int_0^1\int(g_1-g_1')^2e^{-\frac{(v_1\cdot\hat{k}+s\hbar|k|)^2}{8}}|k|\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}s\\
\lesssim&\int_0^1\int_{|k|\geq\frac{1}{\sqrt{\hbar}}}\int g_1^2e^{-\frac{(v_1\cdot\hat{k}+s\hbar|k|)^2}{8}}|k|\hat{\mathcal{V}}(k)^2\,\mathrm{d}v_1\,\mathrm{d}k\,\mathrm{d}s+\int_0^1\int_{|k|\leq\frac{1}{\sqrt{\hbar}}}\int(g_1-g_1')^2e^{-\frac{(v_1\cdot\hat{k}+s\hbar|k|)^2}{8}}\frac{\hat{\mathcal{V}}(k)^2}{\hbar|k|}\,\mathrm{d}k\,\mathrm{d}v_1\\
\lesssim&\hbar\|g\|_\hbar^2.
\end{aligned}$$

***Step* 3.** We need to identify the limit. For $g$ a smooth test function with compact support,

$$\begin{aligned}
\int g_1^2\tilde{K}(v_1)\,\mathrm{d}v_1=&\frac{c_d}{4\hbar^2}\int(g_1^2-g_1'^2)\,(M_2-M_2')\,\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
=&\frac{c_d}{4\hbar^2}\int(g_1''^2-g_1^2)\,(M_2-M_2')\,\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_\hbar(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
=&\frac{c_d}{4}\int k\cdot\nabla g_1^2k\cdot v_2M_2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1+O(\hbar)\\
=&-\frac{c_d}{4}\int g_1^2\mathrm{div}_1\left(\left(\int k\otimes k\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\,\mathrm{d}kM_2\,\mathrm{d}v_2\right)\cdot v_1\right)\mathrm{d}v_1+O(\hbar)
\end{aligned}$$

This conclude the proof. □

3.4. **Dissipation estimates.** We denotes$w_0=\sqrt{M}$, $(w_i)_{i\in[1,d]}=(v\cdot\mathbf{e}_i\sqrt{M})$ (where $(\mathbf{e}_i)_i$ is an orthogonal bases of $\mathbb{R}^d$) and $w_{d+1}:=\frac{|v|^2-d}{\sqrt{2d}}\sqrt{M}$. We introduce $\pi_0$ the $L^2$-orthonormal on $<w_0,\cdots,w_{d+1}>$,

$$\pi_0[g]:=\sum_{i=0}^{d+1}\left(\int w_ig\right)w_i.$$

**Proposition 3.7.** *We fix the dimension $d\geq4$.*

*There exists a constant $C>0$ independant of $\hbar$ and a $\hbar_0>0$ such that for any $\hbar<\hbar_0$, $g\in\mathcal{H}_\hbar$, we have the lower bound*

$$C\int g\mathcal{L}_\hbar g\geq\|\pi_0g\|_\hbar^2. \tag{3.41}$$

*Proof.* We proceed by contradiction. Suppose that there exists a decreasing sequence $\hbar_n \to 0$ and a sequence $(g_n)_n$ such that $\pi_0 g_n = 0$, $\|g_n\|_{\hbar_n} = 1$ and $\int g_n \mathcal{L}_{\hbar_n} g_n \to 0$.

Using Proposition 3.4, we deduce that up to an extraction, the sequence $(g_n)$ converge weakly to $g_\infty$ in all the $\mathcal{H}_\hbar$, and there exists a constant $C$ such that

$$\|g_\infty\|_\hbar^2 < C.$$

**Lemma 3.8.** *Up to the extraction of a subsequence, $g_\infty \in H^1_{\mathrm{loc}}$, $g_n$ converges strongly in $L^2_{\mathrm{loc}}$ and*

$$\|g\|_0^2 \leq 1 \tag{3.42}$$

*Proof.* Fix $K > 0$. We denote $B(K)$ the ball of radius $K$ and of center $x$.

For any $k \in \mathbb{S}^{d-1}$, $\hbar < 0$, and any test function $g$,

$$\begin{aligned}\int_{B(K)} \frac{(g(v_1+\hbar k) - g(v_1))^2}{\hbar^2} \mathrm{d}v_1 &\lesssim \int_{\substack{k \tilde{\in} \frac{k}{2} + B(\frac{1}{4}) \\ v_1 \in B(K)}} \frac{(g(v_1+\hbar k') - g(v_1))^2 + (g(v_1+\hbar k') - g(v_1-k))^2}{2\hbar^2} \mathrm{d}v_1 \,\mathrm{d}k' \\ &\lesssim \int_{\substack{k \tilde{\in} B(\frac{3}{4}) \setminus B(\frac{1}{4}) \\ v_1 \in B(K)}} \frac{(g(v_1+\hbar k') - g(v_1))^2}{\hbar^2} \mathrm{d}v_1 \,\mathrm{d}k' \\ &\lesssim e^{K^2} \frac{1}{\hbar^2} \int (g(v_1) - g(v_1+\hbar k))^2 \frac{\hat{\mathcal{V}}^2(k)}{|k|} e^{-\frac{(v_1\cdot\hat{k})^2}{2}} \mathrm{d}k' \,\mathrm{d}v_1 \leq C_K\end{aligned}$$

for some constant $C_K$ depending on $K$. We deduce the uniform continuity of of the sequence $(g_n)_n$:

$$\sup_n \sup_{|k|=\hbar} \|g_n(k+\cdot) - g_n(\cdot)\|_{L^2(B(K))} \leq C_K \hbar + \max_{n, \hbar_n \geq \hbar} \|g_n(k+\cdot) - g_n(\cdot)\|_{L^2(B(K))}.$$

One can apply Rellich-Kondrakov theorem. Hence the sequence $(g_n)_n$ converges strongly in $L^2(B(K))$.

In addition, as $\|g_\infty\|_\hbar$ is uniformly bounded, for any $k \in \mathbb{R}^d$,

$$\|g_\infty(k+\cdot) - g_\infty\|_{L^2(B(K))} \leq |k| C_K.$$

Hence $g_\infty \in H^1_{\mathrm{loc}}$, and $\frac{(g_\infty(\cdot+\hbar k) - g_\infty(\cdot))}{\hbar}$ converges weakly to $k \cdot \nabla g_\infty$ in $L^2(B(K))$.

For any $K > 0$,

$$\begin{aligned}&\frac{c_d}{\hbar^2} \int_{|v_1|\leq K} \left[ \left(g_{n,1} - g'_{n,1}\right)^2 M_2 + g_{n,1}^2 \left(\sqrt{M_2} - \sqrt{M_2'}\right)^2 \right] \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2 - v_1 - \hbar k)}}{|\varepsilon_\hbar(M,k,v_1)|^2} \mathrm{d}k \,\mathrm{d}v_1 \\ &= \frac{c_d}{\sqrt{2\pi}} \int_{|v_1|\leq K} \left[ \left( \frac{g_n(v_1+\hbar k) - g_n(v_1)}{\hbar} \right)^2 + \frac{g_{n,1}^2}{\hbar^2} \left( 1 - e^{-\frac{2\hbar v_1\cdot k + \hbar^2 |k|^2}{2}} \right)^2 \right] \frac{\hat{\mathcal{V}}(k)^2 e^{-\frac{(v_1\cdot\hat{k})^2}{2}}}{|\varepsilon_\hbar(M,k,v_1)|^2} \mathrm{d}k \,\mathrm{d}v_1 \\ &= \frac{c_d}{\sqrt{2\pi}} \int_{|v_1|\leq K} \left[ \left( \frac{g_n(v_1+\hbar k) - g_n(v_1)}{\hbar} \right)^2 + g_{n,1}^2 (v_1\cdot k)^2 \right] \frac{\hat{\mathcal{V}}(k)^2 e^{-\frac{(v_1\cdot\hat{k})^2}{2}}}{|\varepsilon_0(M,k,v_1)|^2} \mathrm{d}k \,\mathrm{d}v_1 + O\left(\hbar \|g_n\|_\hbar^2\right)\end{aligned}$$

One can take the infimum limit as $n \to 0$:

$$\frac{c_d}{\sqrt{2\pi}} \int_{|v_1|\leq K} \left[ (k\cdot\nabla g_\infty(v_1))^2 + g_\infty(v_1)^2 (v_1\cdot k)^2 \right] e^{-\frac{(v_1\cdot\hat{k})^2}{2}} \frac{\hat{\mathcal{V}}(k)^2}{|\varepsilon_0(M,k,v_1)|^2} \mathrm{d}k \,\mathrm{d}v_1 \leq 1$$

Finally we take the limit $K \to \infty$ by monotone convergence. This conclude the proof. □

In the following, we prove the convergence of $\int g_n \mathcal{L}_{\hbar_n} g_n$ to 0. In dimension 3, we will need a cut-off function $\varphi_K$ introduce in Definition 3.2. If the dimension is bigger than 4, this cut-off function is not needed, and we consider $\varphi_K \equiv 1$.

We recall the decomposition of $\int g_n \mathcal{L}_{\hbar_n} g_n$: for any $K > 0$,

$$\begin{aligned}\int g_n \mathcal{L}_{\hbar_n} g_n =& \frac{c_d}{2{\hbar_n}^2} \int \left[ \left(g_{n,1} - g'_{n,1}\right)^2 M_2 + g'^2_{n,1} \left(\sqrt{M_2} - \sqrt{M_2'}\right)^2 \right] \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2 - v_1 - \hbar_n k)}}{|\varepsilon_{\hbar_n}(M,v_1,k)|^2} \mathrm{d}k \,\mathrm{d}v_1 \,\mathrm{d}v_2 \\ &+ \frac{c_d}{\hbar_n^2} \int \left[ \left(g_{n,1} - g'_{n,1}\right) g'_{n,1} \sqrt{M_2} \left(\sqrt{M_2} - \sqrt{M_2'}\right) \right] \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2 - v_1 - \hbar_n k)}}{|\varepsilon_{\hbar_n}(M,v_1,k)|^2} \mathrm{d}k \,\mathrm{d}v_1 \,\mathrm{d}v_2 \\ &+ \frac{c_d}{2\hbar_n^2} \int \left( g_{n,1}\sqrt{M_2} - g'_{n,1}\sqrt{M_2'} \right) \left( g_{n,2}\sqrt{M_1} - g'_{n,2}\sqrt{M_1'} \right) \frac{\hat{\mathcal{V}}(k)^2 \delta_{k\cdot(v_2 - v_1 - \hbar k)}}{|\varepsilon_\hbar(M,v_1,k)|^2} \mathrm{d}k \,\mathrm{d}v_1 \,\mathrm{d}v_2\end{aligned}$$

- Using, the first line is equal to

$$\frac{c_d}{2{\hbar_n}^2}\int\left[\left(g_{n,1}-g'_{n,1}\right)^2 M_2+g'^2_{n,1}\left(\sqrt{M_2}-\sqrt{M'_2}\right)^2\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar_n k)}}{|\varepsilon_{\hbar_n}(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2=\frac{\|g_n\|_{\hbar_n}}{2}=\frac{1}{2}.$$

- For second line, using Proposition 3.6 and that

$$\int_{|v_1|>K} g_n^2(v_1)\left|\tilde{\mathcal{K}}_0(v_1)\right|\mathrm{d}v_1\lesssim\int_{|v_1|>K}\frac{g_n(v_1)^2}{\langle v_1\rangle^3}\,\mathrm{d}v_1=O(\tfrac{1}{K}),$$

one have

$$\frac{c_d}{\hbar_n^2}\int\left[\left(g_{n,1}-g'_{n,1}\right)g'_{n,1}\sqrt{M_2}\left(\sqrt{M_2}-\sqrt{M'_2}\right)\right]\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar_n k)}}{|\varepsilon_{\hbar_n}(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$=\int_{|v_1|\le K} g_n^2(v_1)\tilde{\mathcal{K}}_0(v_1)\,\mathrm{d}v_1+O\left(\tfrac{1}{K}+\hbar_n\right).$$

- For the third line, using Proposition 3.5, we perform the decomposition

$$\frac{c_d}{\hbar_n^2}\int\left(g_{n,1}\sqrt{M_2}-g'_{n,1}\sqrt{M'_2}\right)\left(g_{n,2}\sqrt{M_1}-g'_{n,2}\sqrt{M'_1}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar_n k)}}{|\varepsilon_{\hbar_n}(M,v_1,k)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$=\int_{\substack{|v_1|<K\\|v_2|<K}}\left(\varphi_K(v_1-v_2)\mathcal{K}_{\hbar_n}(v_1,v_2)-\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)g_n(v_1)g_n(v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2+O\left(\tfrac{1}{K}+K^3\hbar_n\right).$$

We used that

$$\int_{\{|v_1|\ge K\}}\varphi_K(v_1-v_2)\left(|\mathcal{K}_{\hbar_n}(v_1,v_2)|+\left|\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right|\right)|g_{n,1}|\,|g_{n,2}|\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$\lesssim\int_{\substack{|v_1|\ge K\\|v_1-v_2|>\frac{1}{3K}}}\frac{\sqrt[5]{M_1M_2}}{|v_1-v_2|^3}|g_{n,1}|\,|g_{n,2}|\,\mathrm{d}v_1\,\mathrm{d}v_2=O(\tfrac{1}{K}).$$

We can take now the limit $n\to\infty$, using that $g_n$ converges strongly to $g_\infty$ in $L^2_{\mathrm{loc}}$:

$$0=\frac{1}{2}+\frac{1}{2}\int_{\substack{|v_1|<K\\|v_2|<K}}\left(\varphi_K(v_1-v_2)\mathcal{K}_0(v_1,v_2)-\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)g_\infty(v_1)g_\infty(v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$+\int_{|v_1|\le K} g_\infty(v_1)^2\tilde{\mathcal{K}}_0(v_1)\,\mathrm{d}v_1+O(\tfrac{1}{K})$$
$$\ge\frac{\|g_\infty\|_0^2}{2}+\frac{1}{2}\int\left(\varphi_K(v_1-v_2)\mathcal{K}_0(v_1,v_2)-\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)g_\infty(v_1)g_\infty(v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 \tag{3.43}$$
$$+\int g_\infty(v_1)^2\tilde{\mathcal{K}}_0(v_1)\,\mathrm{d}v_1+O(\tfrac{1}{K}).$$

One can perform an integration by part and

$$\int g_\infty(v_1)^2\tilde{\mathcal{K}}_0(v_1)\,\mathrm{d}v_1=\frac{c_d}{4}\int g_\infty(v_1)^2\nabla_1\cdot\left(v_1\int k\otimes k\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(M,k,v_1)|^2}\,\mathrm{d}kM_2\,\mathrm{d}v_2\right)$$
$$=-\frac{c_d}{2}\int\nabla g_\infty(v_1)B(v_1,v_2)(g_\infty(v_1)v_1)M_2\,\mathrm{d}v_2\,\mathrm{d}v_1\,\mathrm{d}v_2.$$

In the same way, using that $B(v_1,v_2)(v_1-v_2)=0$,

$$\left(\nabla_1-\frac{v_1}{2}\right)\otimes\left(\nabla_2-\frac{v_2}{2}\right):\left(\varphi_K(v_1-v_2)\sqrt{M_1M_2}B(v_1,v_2)\right)+\varphi_K(v_1-v_2)\mathcal{K}_0(v_1-v_2)$$
$$=-\nabla^2\varphi_K(v_1-v_2):B(v_1,v_2)\sqrt{M_1M_2}+\nabla\varphi_K(v_1-v_2)\otimes\left(\nabla_2-\nabla_1-\tfrac{v_2-v_1}{2}\right):\left(\sqrt{M_1M_2}B(v_1,v_2)\right)$$
$$=\sqrt{M_1M_2}\left(-\nabla^2\varphi_K(v_1-v_2):B(v_1,v_2)-\nabla\varphi_K(v_1-v_2)\cdot((\nabla_1-\nabla_2)\cdot B(v_1,v_2))\right)$$

Using that $\varphi_K(v)=\varphi(K|v|)$,

$$\nabla^2\varphi_K(v_1-v_2):B(v_1,v_2)=\frac{K\varphi'(K|v_1-v_2|)}{|v_1-v_2|^2}\int_{<v_1-v_2>^\perp}\frac{|\hat{\mathcal{V}}(k)|^2|k|^2\,\mathrm{d}k}{|\varepsilon_0(M,\frac{v_1+v_2}{2},k)|^2}=\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2}),$$

where $\rho_K$ has been introduce in (3.21).

$$\begin{aligned}
&\mathrm{div}_{v_1}\int \frac{\hat{\mathcal{V}}(k)^2 k\otimes k}{|\varepsilon_0(M,k,v_1)|^2}\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k\\
&=\int k\cdot\nabla_{v_1}\left(\frac{\hat{\mathcal{V}}(k)^2}{|\varepsilon_0(M,k,v_1)|^2}\right)k\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k+\int\frac{\hat{\mathcal{V}}(k)^2|k|^2}{|\varepsilon_0(M,k,v_1)|^2}k\delta'_{k\cdot(v_1-v_2)}\,\mathrm{d}k\\
&=\int k\cdot\nabla_{v_1}\left(\frac{\hat{\mathcal{V}}(k)^2}{|\varepsilon_0(M,k,v_1)|^2}\right)k\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k-\int\frac{v_1-v_2}{|v_1-v_2|^2}\cdot\nabla_k\left(\frac{\hat{\mathcal{V}}(k)^2|k|^2}{|\varepsilon_0(M,k,v_1)|^2}\right)k\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k\\
&\qquad-\frac{v_1-v_2}{|v_1-v_2|^2}\int\frac{\hat{\mathcal{V}}(k)^2|k|^2}{|\varepsilon_0(M,k,v_1)|^2}\delta_{k\cdot(v_1-v_2)}\,\mathrm{d}k.
\end{aligned}$$

Using that $\nabla\varphi_K(v)=K\varphi'(K|v|)\frac{v}{|v|}$, one have

$$-\nabla\varphi_K(v_1-v_2)\cdot((\nabla_1-\nabla_2)\cdot B(v_1,v_2))=2\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2}).$$

Finally,

$$\begin{aligned}
&\int\left(\varphi_K(v_1-v_2)\mathcal{K}_0(v_1,v_2)-\rho_K(v_1-v_2,\tfrac{v_1+v_2}{2})\right)g_\infty(v_1)g_\infty(v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2\\
&=-\int g_\infty(v_1)g_\infty(v_2)\left(\nabla_1-\frac{v_1}{2}\right)\otimes\left(\nabla_2-\frac{v_2}{2}\right):\left[\varphi_K(v_1-v_2)B(v_1,v_2)\sqrt{M_1M_2}\right]\mathrm{d}v_1\,\mathrm{d}v_2\\
&=-\int\varphi_K(v_1-v_2)\left(\nabla_1+\frac{v_1}{2}\right)g_\infty(v_1)B(v_1,v_2)\left(\nabla_2+\frac{v_2}{2}\right)g_\infty(v_2)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2\\
&\quad+\int(1-\varphi_K(v_1-v_2))\left(\nabla_1+\frac{v_1}{2}\right)g_\infty(v_1)B(v_1,v_2)\left(\nabla_2+\frac{v_2}{2}\right)g_\infty(v_2)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2.
\end{aligned}$$

One can bound the est using the Young inequality:

$$\begin{aligned}
&\int|1-\varphi_K(v_1-v_2)|\left|\left(\nabla_1+\frac{v_1}{2}\right)g_\infty(v_1)\right||B(v_1,v_2)|\left|\left(\nabla_2+\frac{v_2}{2}\right)g_\infty(v_2)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2\right|\\
&\qquad\lesssim\left\|\frac{1-\varphi_K(v)}{|v|}\right\|_{L^1}\left\|\left(\nabla_2+\frac{v_2}{2}\right)g_\infty(v_2)\sqrt{M_2}\right\|_{L^2}^2=O(\tfrac{1}{K}).
\end{aligned}$$

Hence, taking the limit $K\to\infty$ in (3.43),

$$\begin{aligned}
0\geq\frac{1}{2}\int&\left[\left(\nabla_1+\frac{v_1}{2}\right)g_{\infty,1}\sqrt{M_2}-\left(\nabla_2+\frac{v_2}{2}\right)g_{\infty,2}\sqrt{M_1}\right]B(v_1,v_2)\\
&\left[\left(\nabla_1+\frac{v_1}{2}\right)g_{\infty,1}\sqrt{M_2}-\left(\nabla_2+\frac{v_2}{2}\right)g_{\infty,2}\sqrt{M_1}\right]\mathrm{d}v_1\,\mathrm{d}v_2
\end{aligned}$$

As this term is non-negative, it is zero. We deduce that (see [DW23], Subset 2.2 of the proof of Proposition 2.5), $g_\infty\in<w_0,\cdot,w_{d+1}>$, which contradict the fact that $\pi_0(g_\infty)=0$. This conclude the proof. ☐

**Proposition 3.9.** *For any test function $g$*

$$\left\|\pi_0\left[\left(\nabla-\frac{v}{2}\right)g\right]\right\|_\hbar\lesssim\min(\|g\|_\hbar,\|g\|) \tag{3.44}$$

*Proof.* We recall that

$$\pi_0[g]:=\sum_{i=0}^{d+1}\left(\int w_ig\right)w_i$$

As $|wi|=O(\sqrt[4]{M})$ and $\|w_i\|_\hbar\lesssim\|w_i\|_0<\infty$, we can conclude. ☐

3.5. **Non-linear estimation and proof of Theorem.** In the following, we denote $\mathcal{H}^r_\hbar$ the Hilbert space of norm

$$\|g\|^2_{\hbar,r}:=\sum_{r'=0}^{r}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g\right\|_\hbar^2.$$

The present section is dedicated to bound $\mathcal{L}_\hbar$ and $\mathcal{Q}_\hbar$.

We denote

$$\Lambda(p)(v_1,k):=\int\frac{p(v_3)\sqrt{M(v_3)}-p(v_3+\hbar k)\sqrt{M(v_3+\hbar k)}}{\hbar k\cdot(v_3-v_1-\hbar k)+i0}\,\mathrm{d}v_3 \tag{3.45}$$

**Proposition 3.10.** *We fix the dimension $d \geq 2$. The two following bounds hold:*

$$\int \left|\Delta\!\!\!\Delta\left(f\sqrt{M_*}\right)\right| |\Delta\!\!\!\Delta\,(gh_*)| \frac{\hat{\mathcal{V}}^2(k)\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 \lesssim \|f\|_\hbar\|g\|_\hbar\|h\|_3 \tag{3.46}$$

$$\int \left|\Delta\!\!\!\Delta\left(f\sqrt{M_*}\right)\right| |\Delta\!\!\!\Delta\,(gh_*)|\,|\Lambda(p)| \frac{\hat{\mathcal{V}}^3(k)\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 \lesssim \|f\|_\hbar\|p\|_\hbar\|g\|_2\|h\|_2 \tag{3.47}$$

*Proof.* We begin by (3.46). Using the symmetry of the expression, it is sufficient to bound

$$\frac{1}{\hbar^2}\int\left|f_1\sqrt{M_2}-f_1'\sqrt{M_2'}\right| |g_1h_2-g_1'h_2'|\,\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1$$

We can split it into four parts

$$\begin{aligned}
&\frac{1}{\hbar^2}\int\left|e^{\frac{|v_2\cdot k|^2}{8|k|^2}}(f_1-f_1')\sqrt{M_2}\right|\left|e^{-\frac{|v_2\cdot k|^2}{8|k|^2}}g_1(h_2-h_2')\right|\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
&+\frac{1}{\hbar^2}\int\left|e^{\frac{|v_2\cdot k|^2}{8|k|^2}}f_1'(\sqrt{M_2'}-\sqrt{M_2})\right|\left|e^{-\frac{|v_2\cdot k|^2}{8|k|^2}}g_1(h_2-h_2')\right|\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
&+\frac{1}{\hbar^2}\int\left|(f_1-f_1')\sqrt[4]{M_2}\right|\left|\sqrt[4]{M_2}(g_1-g_1')h_2'\right|\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
&+\frac{1}{\hbar^2}\int\left|f_1'(\sqrt[4]{M_2}-\sqrt[4]{M_2'})\right|\left|(\sqrt[4]{M_2}+\sqrt[4]{M_2'})(g_1-g_1')h_2'\right|\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1
\end{aligned}$$

We apply then the Cauchy-Schwartz inequality to each term of the sum.

$$\int\left[e^{\frac{|v_2\cdot k|^2}{8|k|^2}}(f_1-f_1')\sqrt{M_2}\right]^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1}{\hbar^2}=\int(f_1-f_1')^2\frac{\hat{\mathcal{V}}^2(k)}{\hbar^2|k|}e^{\frac{|v_1\cdot k|^2}{8|k|^2}}\,\mathrm{d}k\,\mathrm{d}v_1\lesssim\|f\|_\hbar^2$$

$$\int\left[e^{\frac{|v_2\cdot k|^2}{8|k|^2}}f_1'(\sqrt{M_2'}-\sqrt{M_2})\right]^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1}{\hbar^2}\lesssim\|f\|_\hbar^2$$

$$\begin{aligned}
&\int e^{-\frac{|v_2\cdot k|^2}{4|k|^2}}g_1^2(h_2-h_2')^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1}{\hbar^2}\\
&\qquad\lesssim\frac{\|g\|^2_{L^\infty(<\hat{k}>,L^2(<\hat{k}>^\perp))}}{\hbar^2}\int e^{-\frac{|v_2\cdot k|^2}{4|k|^2}}(h_2-h_2')^2\frac{\hat{\mathcal{V}}(k)^2}{|k|}\,\mathrm{d}k\,\mathrm{d}v_2\lesssim\|g\|_2^2\|h\|_\hbar^2
\end{aligned}$$

$$\begin{aligned}
&\int e^{-\frac{|v_2\cdot k|^2}{4|k|^2}}g_1^2(h_2-h_2')^2\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1}{\hbar^2}\\
&\qquad\lesssim\|\nabla h\|^2_{L^\infty(<\hat{k}>,L^2(<\hat{k}>^\perp))}\int g_1^2e^{-\frac{|v_2\cdot k|^2}{4|k|^2}}|k|^2\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1\leq\|h\|_2^2\|g\|_\hbar^2
\end{aligned}$$

We treat now (3.47). We bound first $\Lambda$: using the $L^2\to L^2$ bound on the Hilbert transform, denoting $v_3:=v_1^\parallel+v_3^\perp$ and $v_3'=v_3+\hbar k$,

$$\begin{aligned}
\int\frac{\hat{\mathcal{V}}^2(k)}{|k|}|\Lambda(p)(v_1,k)|^2\,\mathrm{d}v_1^\parallel\,\mathrm{d}k&\leq\int\frac{\hat{\mathcal{V}}^2(k)}{|k|}\left|\int_{<k^\perp>}(p_1\sqrt{M_1}-p_1'\sqrt{M_1})\right|^2\mathrm{d}v_1^\parallel\,\mathrm{d}k\\
&\leq\int\frac{\hat{\mathcal{V}}^2(k)}{|k|}\left(\int_{<k^\perp>}(p_1-p_1')^2\,\mathrm{d}v_1^\perp\right)\left(\int_{<k^\perp>}M_3\,\mathrm{d}v_3\right)\mathrm{d}v_1^\parallel\,\mathrm{d}k\\
&\quad+\int\frac{\hat{\mathcal{V}}^2(k)}{|k|}\left(\int_{<k^\perp>}{p_1'}^2\,\mathrm{d}v_1^\perp\right)\left(\int_{<k^\perp>}(\sqrt{M_3}-\sqrt{M_3'})^2\,\mathrm{d}v_3\right)\mathrm{d}v_1^\parallel\,\mathrm{d}k\lesssim\hbar^2\|p\|_\hbar^2
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\hbar^2}\int\left[\Lambda(p)(h_1g_2-h_1'g_2')\hat{\mathcal{V}}(k)^2\right]^2\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1\\
&\qquad\lesssim\|p\|_\hbar^2\|g\|^2_{L^\infty(<\hat{k}>,L^2(<\hat{k}>^\perp))}\|h\|^2_{L^\infty(<\hat{k}>,L^2(<\hat{k}>^\perp))}\lesssim\|p\|_\hbar^2\|g\|_1\|h\|_1.
\end{aligned}\tag{3.48}$$

□

**Proposition 3.11.** *We fix the dimension $d\geq 2$. Denoting $\tilde{r}:=\max(5,r)$, for $f,g,h$ three tests functions*

$$\left|\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\mathcal{Q}_\hbar(g,h)\right|\lesssim(1+\|h\|_{\tilde r-1})^{r+1}\left(\|g\|_{\hbar,r}\|h\|_{\tilde r-1}+\|g\|_{\hbar,r}\|h\|_{\tilde r-1}\right)\|\,\|f\|_{\hbar,r}$$
$$\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\mathcal{L}_\hbar g=\int\left(\nabla-\frac{v}{2}\right)^r f\mathcal{L}_\hbar\left(\nabla-\frac{v}{2}\right)^r g+O\left(\|g\|_{\hbar,r-1}\|\,\|f\|_{\hbar,r}\right)$$
$$\left|\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\left[\mathcal{Q}_\hbar(g,h)-\mathcal{Q}_\hbar(\tilde g,\tilde h)\right]\right|$$
$$\lesssim\left(1+\|h\|_{\tilde r-1}+\|\tilde h\|_{\tilde r-1}\right)^{r+1}\left(\|g-\tilde g\|_{\hbar,r}+\|g-\tilde g\|_{\tilde r-1}\left(\|h\|_{\hbar,r}+\|\tilde h\|_{\hbar,r}\right)\right.$$
$$\left.+\left(\|g\|_{\hbar,r}+\|\tilde g\|_{\hbar,r}\right)\|h-\tilde h\|_{\tilde r-1}+\left(\|g\|_{\tilde r-1}+\|\tilde g\|_{\tilde r-1}\right)\|h-\tilde h\|_{\hbar,r}\right)\|f\|_{\hbar,r}$$

*Proof.* As we want perform integration by part, it will be useful to introduce the change of variable $k\mapsto\sigma$ where

$$k=:\frac{|v_1-v_2|}{2\hbar}\left[\sigma-\frac{v_1-v_2}{|v_1-v_2|}\right],$$

with the Jacobian

$$\hbar^{-2}\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k=4\hbar^{d-1}|v_2-v_1|^{d-2}\delta_{|\sigma|=1}\,\mathrm{d}\sigma.$$

With this variables,

$$v_1'=\frac{v_1+v_2}{2}+\frac{|v_1-v_2|}{2}\sigma,\;\; v_2'=\frac{v_1+v_2}{2}-\frac{|v_1-v_2|}{2}\sigma.$$

$$\left(\nabla_{v_1}+\nabla_{v_2}\right)k(v_1,v_2,\sigma)=\frac{1}{2\hbar}\left(\frac{(v_1-v_2)\cdot\sigma}{|v_1-v_2|}-\mathrm{Id}+\frac{(v_2-v_1)\cdot\sigma}{|v_2-v_1|}+\mathrm{Id}\right)=0.$$

Hence $(\nabla_1+\nabla_2)g(v_1')=\nabla g(v_1')$ and

$$⩍\left(\frac{\left(\nabla+\frac{v}{2}\right)g}{\sqrt{M}}\right)=⩍\left(\nabla\frac{g}{\sqrt{M}}\right)=(\nabla_1+\nabla_2)⩍\left(\frac{g}{\sqrt{M}}\right)$$

$$(\nabla_1+\nabla_2)⩍\left(gh_*\sqrt{MM_*}\right)=⩍\left(\left[\left(\nabla-\frac{v}{2}\right)gh_*+g\left(\nabla_*-\frac{v_*}{2}\right)h_*\right]\sqrt{MM_*}\right)$$

For $f,g,h,F=M+\sqrt{M}p$

$$(-1)^r\int ⩍\left(\frac{\left(\nabla+\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f}{\sqrt{M}}\right)\frac{\hat{\mathcal{V}}(k)^2|v_1-v_2|^{d-2}}{|\varepsilon(F,k,v_1)|^2}⩍\left(gh_*\right)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}\sigma$$
$$=(-1)^r\int(\nabla_1+\nabla_2)^r⩍\left(\frac{\left(\nabla-\frac{v}{2}\right)^r f}{\sqrt{M}}\right)\frac{\hat{\mathcal{V}}(k)^2|v_1-v_2|^{d-2}}{|\varepsilon(F,k,v_1)|^2}⩍\left(gh_*\right)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}\sigma$$
$$=\sum_{r_1+r_2+r_3=r}\binom{r}{r_1,r_2,r_3}\int ⩍\left(\frac{\left(\nabla-\frac{v}{2}\right)^r f}{\sqrt{M}}\right)\hat{\mathcal{V}}(k)^2|v_1-v_2|^{d-2}\frac{\partial^{r_1}|\varepsilon(F,k,v_1)|^{-2}}{(\partial v_1)^{r_1}}$$
$$\times ⩍\left(\left(\nabla-\frac{v}{2}\right)^{r_2}g\left(\nabla_*-\frac{v_*}{2}\right)^{r_3}h_*\right)\sqrt{M_1M_2}\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}\sigma$$

Using the Faà di Bruno formula, and denoting $\mathfrak{c}(z)=\overline{z}$ the convolution and $F=M+\sqrt{M}p$, $\tilde F=M+\sqrt{M}\tilde p$

$$\left|\frac{\partial^{\gamma_j}|\varepsilon(F,k,v_1)|^{-2}}{(\partial v_1)^{\gamma_j}}-\frac{\partial^{\gamma_j}|\varepsilon(\tilde F,k,v_1)|^{-2}}{(\partial v_1)^{\gamma_j}}\right|\lesssim_r\sum_{n=1}^{r_1}\sum_{\substack{\gamma_1+\dots+\gamma_n\\ \gamma_i>0\\ \sigma_i=\pm1}}\left|\frac{\prod_{j=1}^n\mathfrak{c}^{\sigma_i}\frac{\partial^{\gamma_j}\varepsilon(F,k,v_1)}{(\partial v_1)^{\gamma_j}}}{|\varepsilon_\hbar(F,k,v_1)|^{r_1+2}}-\frac{\prod_{j=1}^n\mathfrak{c}^{\sigma_i}\frac{\partial^{\gamma_j}\varepsilon(\tilde F,k,v_1)}{(\partial v_1)^{\gamma_j}}}{|\varepsilon_\hbar(\tilde F,k,v_1)|^{r_1+2}}\right|$$

We recall that

$$\frac{\partial^{\gamma_j}\varepsilon(F,k,v)}{(\partial v_1)^{\gamma_j}}-\frac{\partial^{\gamma_j}\varepsilon(\tilde F,k,v)}{(\partial v_1)^{\gamma_j}}=\hat{\mathcal{V}}(k)\int\frac{\nabla^{\gamma_j}\left[\sqrt{M}(p-\tilde p\right](v_*)-\nabla^{\gamma_j}\left[\sqrt{M}(p-\tilde p)\right](v_*-\hbar k)}{\hbar k\cdot(v-v_*-\hbar k)+i0}\mathrm{d}v_*$$
$$=\hat{\mathcal{V}}(k)\Lambda\left(\left(\nabla-\frac{v}{2}\right)^{\gamma_j}(p-\tilde p)\right)$$

As for any function $p$,

$$\left\|\hat{\mathcal{V}}(k)\Lambda(p)\right\|_{L^\infty(\mathrm{d}k\,\mathrm{d}v_1)} \lesssim \left\|\hat{\mathcal{V}}(k)\nabla_{v_1}\Lambda(p)\right\|_{L^\infty(\mathrm{d}kL^2(\mathrm{d}v_1\cdot\hat{k}))} = \left\|\hat{\mathcal{V}}(k)\Lambda\left(\left(\nabla-\frac{v}{2}\right)p\right)\right\|_{L^\infty(\mathrm{d}k,L^2(\mathrm{d}v_1\cdot\hat{k}))}$$
$$\lesssim \left\|\frac{\hat{\mathcal{V}}(k)}{\hbar|k|}\int\left(\left(\nabla-\frac{v}{2}\right)p(v_*+\hbar k)-\left(\nabla-\frac{v}{2}\right)p(v_*)\right)\mathrm{d}v_*^\perp\right\|_{L^\infty(\mathrm{d}k,L^2(\mathrm{d}v_1\cdot\hat{k}))} \lesssim \|p\|_2\,.$$

We deduce that

$$\left|\frac{\partial^r|\varepsilon(F,k,v_1)|^{-2}}{(\partial v_1)^{\gamma_j}}-\frac{\partial^r|\varepsilon(\tilde{F},k,v_1)|^{-2}}{(\partial v_1)^r}\right|$$
$$\lesssim_r |\hat{\mathcal{V}}(k)|\left(1+\|p\|_{\max(4,r-1)}+\|\tilde{p}\|_{\max(4,r-1)}\right)^r\left(\sum_{s=0}^{2}\left|\Lambda\left(\left(\nabla-\frac{v}{2}\right)^{r-r}(p-\tilde{p})\right)\right|+\|\tilde{p}-p\|_{r-1}\right).$$

Using that $r_1+r_2+r_3=r$, and Proposition 3.10 denoting $\tilde{r}:=\max\{5,r\}$

$$\int \mathbb{\Delta}\left(\frac{\left(\nabla-\frac{v}{2}\right)^r\left(-\nabla-\frac{v}{2}\right)^r f}{\sqrt{M}}\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2}\left[|\varepsilon(F,k,v_1)|^{-2}-|\varepsilon(\tilde{F},k,v_1)|^{-2}\right]$$
$$\times\mathbb{\Delta}\left(\frac{gh_*}{\sqrt{MM_*}}\right)M_1M_2\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}k$$
$$\lesssim \left(1+\|p\|_{\tilde{r}-1}+\|\tilde{p}\|_{\tilde{r}-1}\right)^{r+1}\|f\|_{\hbar,r}\left(\|g\|_{\tilde{r}-1}\|h\|_{r,\hbar}+\|h\|_{\tilde{r}-1}\|g\|_{r,\hbar}+\|g\|_{\tilde{r}-1}\|h\|_{\tilde{r}-1}\|p-\tilde{p}\|_{r,\hbar}\right)$$

One can conclude by choosing suitable $p$ and $h$. □

We introduce the space $\mathcal{C}^0_b(\mathbb{R}^+,\mathcal{H}_r)\cap L^2(\mathbb{R}^+,\mathcal{H}^r_\hbar)$, and define the sets

$$\mathcal{E}_{r,\hbar}(\kappa,\eta)=\left\{h\in\mathcal{H}^r_\hbar\middle|\|h(t=0)\|_r\leq\kappa\eta,\ \|h\|_{L^\infty_t(\mathcal{H}^r)}+\|h\|_{L^2(\mathcal{H}^r_\hbar)}\leq\eta\right\}$$
$$\mathcal{E}_{r,\hbar}(\kappa,\eta,h_0)=\{h\in\mathcal{E}_\hbar(\eta),\ h(t=0)=h_0\}$$

where $\kappa_r\in(0,1)$ is fix constant depending only on $r$ and $\mathcal{V}$, and $\|h_0\|_r\leq\kappa_r\eta$. The constant $\kappa_r$ will be fixed later.

**Proposition 3.12.** *We fix the dimension $d\geq 2$. We define $\mathcal{F}_\hbar$ the application which associate to $h\in\mathcal{E}_{r,\hbar}$ the solution $g$ of the equation*

$$\begin{cases}\partial_t g(t)+\mathcal{L}_\hbar g(t)=\mathcal{Q}_\hbar(g(t),h(t))\\ g(t=0)=h(t=0).\end{cases}\tag{3.49}$$

*Then for $r>5$, there exist three constants $\kappa_r\in(0,1)$, $\eta_0>0$, $\hbar_0>0$ such that for $\kappa\in(0,\kappa_r)$, $\eta\in(0,\eta_0)$, and $\hbar\in(0,\hbar_0)$,*

- *the sets $\mathcal{E}_{r,\hbar}(\kappa,\eta)$ and $\mathcal{E}_{r,\hbar}(\kappa,`\eta,h_0)$ are stable under the action of $\mathcal{F}_\hbar$,*
- *the application $\mathcal{F}_\hbar$ is continuous on $\mathcal{E}_{r,\hbar}(\kappa,\eta)$ and contractive on $\mathcal{E}_{r,\hbar}(\kappa,\eta,h_0)$.*

*Proof.* ***Step* 1.** The stability of $\mathcal{E}(\eta)$.

Fix $h\in\mathcal{E}_{r,\hbar}(\kappa,\eta)$ and $g:=\mathcal{F}_\hbar(g)$. We have the following energy inequality.

$$\frac{d}{dt}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|^2=2\int\left(-\nabla-\frac{v}{2}\right)^{r'}\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\left(-\mathcal{L}_\hbar g(t)+\mathcal{Q}_\hbar(g(t),h(t))\right)\mathrm{d}$$
$$\leq-2\int\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\mathcal{L}_\hbar\left(\nabla-\frac{v}{2}\right)^{r'}g(t)+C\|g(t)\|^2_{\hbar,r'-1}$$
$$+C\left(1+\|h(t)\|_{\tilde{r}-1}\right)^{r'+1}\left(\|g(t)\|_{\hbar,r'}\|h(t)\|_{r'-1}+\|g(t)\|_{r'-1}\|h(t)\|_{\hbar,r'}\right)\|\,\|g(t)\|_{\hbar,r}$$
$$\leq-\frac{2}{C}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|^2_\hbar+C\|g(t)\|^2_{r'-1}$$
$$+C\left(1+\|h(t)\|_{\tilde{r}-1}\right)^{r'+1}\left(\|g(t)\|_{\hbar,r'}\|h(t)\|_{r'-1}+\|g(t)\|_{r'-1}\|h(t)\|_{\hbar,r'}\right)\|\,\|g(t)\|_{\hbar,r}$$

where we use that $\pi_0(g(t))=0$, and Propositions 3.7 and 3.9.

We deduce that

$$\frac{d}{dt}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|^2+\left(\frac{1}{C}-C\left(1+\|h(t)\|_{\tilde r-1}\right)^{r+1}\left(\|h(t)\|_{\tilde r-1}+\|g(t)\|_{\tilde r-1}\right)\right)\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|_\hbar^2$$
$$\leq C\left(1+\|h(t)\|_{\tilde r-1}\right)^{r'+1}\|g(t)\|_{\tilde r-1}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}h(t)\right\|_\hbar^2+C\|g(t)\|^2_{\hbar,r'-1}$$

Summing on $r'$,

$$\frac{d}{dt}\sum_{r'=0}^{r}\frac{2^{r'}}{C^{r'}}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|^2$$
$$+\left(\frac{1}{2C}-C\left(1+\|h(t)\|_{\tilde r-1}\right)^{r+1}\left(\|h(t)\|_{\tilde r-1}+\|g(t)\|_{\tilde r-1}\right)\right)\sum_{r'=0}^{r}\frac{2^{r'}}{C^{r'}}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}g(t)\right\|_\hbar^2$$
$$\leq C\left(1+\|h(t)\|_{\tilde r-1}\right)^{r'+1}\|g(t)\|_{\tilde r-1}\sum_{r'=0}^{r}\frac{2^{r'}}{C^{r'}}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}h(t)\right\|_\hbar^2$$

Let $T>0$ such that for all $t\in[0,T)$, $\|h(t)\|_r\leq\eta$. One can integrate the previous inequality: for $t\in[0,T)$

$$\|g(t)\|_r^2+\left(\frac{1}{2C}-2C2^{r+1}\eta\right)\int_0^t\|g(s)\|^2_{\hbar,r}\,\mathrm{d}s\leq\frac{C^r}{2^r}\left(\|g(0)\|_r^2+C2^{r+1}\eta\int_0^t\|g(s)\|^2_{\hbar,r}\,\mathrm{d}s\right)$$

Finally, for $\eta<\eta_0:=(4C^{2r+2})^{-1}$, $\kappa_r=C^{-(r+1)/2}$, we obtain

$$\|g(t)\|_r^2+\frac{1}{4C}\int_0^t\|g(s)\|^2_{\hbar,r}\,\mathrm{d}s\leq\frac{1}{4C}\eta^2$$

By a boot-strap argument, we deduce that $g\in\mathcal{E}_\hbar(\eta)$.

***Step* 2.** Continuity in time of $g$:
As $\|g(t)\|_{r,\hbar}$ is a $L^2(\mathbb{R}^+)$, it is bounded for almost all $t'\in\mathbb{R}^+$.

$$\frac{d}{dt}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-g(t'))\right\|^2\leq-2\int\left(-\nabla-\frac{v}{2}\right)^{r'}\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-g(t'))\,\mathcal{L}_\hbar\left(g(t)-g(t')\right)$$
$$-2\int\left(-\nabla-\frac{v}{2}\right)^{r'}\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-g(t'))\,\mathcal{L}_\hbar g(t')$$
$$+C\left(1+\|h(t)\|_{\tilde r-1}\right)^{r'+1}\left(\|g(t)\|_{\hbar,r'}\|h(t)\|_{r'-1}+\|g(t)\|_{r'-1}\|h(t)\|_{\hbar,r'}\right)\|\,\|g(t)-g(t')\|_{\hbar,r}$$
$$\leq-\frac{1}{2C}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-g(t'))\right\|_\hbar^2+4C^2\|g(t')\|^2_{\hbar,r}+C\|g(t)-g(t')\|^2_{\hbar,r'-1}$$
$$+\tilde C\eta^2\left(\|h(t)\|^2_{\hbar,r}+\|h(t)\|^2_{\hbar,r}\right)$$

Summing on $r'\in[0,r]$, we get

$$\frac{d}{dt}\|g(t)-g(t')\|_r^2\lesssim\|g(t')\|^2_{\hbar,r}+\eta^2\left(\|h(t)\|^2_{\hbar,r}+\|h(t)\|^2_{\hbar,r}\right).$$

We deduce that $g$ is continuous for almost all $t'\in\mathbb{R}^+$, and so on $\mathbb{R}^+$.

***Step* 3.** Continuity and contractility of $\mathcal{F}_\hbar$.

$$\frac{d}{dt}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-\tilde g(t))\right\|^2=-2\int\left(-\nabla-\frac{v}{2}\right)^{r'}\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-\tilde g(t))\,\mathcal{L}_\hbar\left(g(t)-\tilde g(t)\right)$$
$$+2\int\left(-\nabla-\frac{v}{2}\right)^{r'}\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-\tilde g(t))\left(\mathcal{Q}_\hbar(g(t),h(t))-\mathcal{Q}_\hbar(\tilde g(t),\tilde h(t))\right)$$
$$\leq-\frac{1}{C}\left\|\left(\nabla-\frac{v}{2}\right)^{r'}(g(t)-\tilde g(t))\right\|_\hbar+C\left\|(g(t)-\tilde g(t))\right\|^2_{\hbar,r'-1}$$
$$+C\left(1+\|h(t)\|_{\tilde r}+\|\tilde h(t)\|_{\tilde r}\right)^{r+1}\Big(\left(\|h\|_{\tilde r}+\|\tilde h\|_{\tilde r}\right)\|g(t)-\tilde g(t)\|_{\hbar,r}+\|g-\tilde g\|_{\tilde r}\left(\|h\|_{\hbar,r}+\|\tilde h\|_{\hbar,r}\right)$$
$$+\left(\|g\|_{\hbar,r}+\|\tilde g\|_{\hbar,r}\right)\|h(t)-\tilde h(t)\|_{\tilde r}+\left(\|g\|_{\tilde r}+\|\tilde g\|_{\tilde r}\right)\|h(t)-\tilde h(t)\|_{\hbar,r}\Big)\|g(t)-\tilde g(t)\|_{\hbar,r}$$

Summing on $r' \in [0,r]$ and using the polarization inequalities, and that $g, \tilde g, h, \tilde h \in \mathcal{E}_\hbar(\eta)$

$$\frac{d}{dt}\|g(t)-\tilde g(t)\|_r^2 + \|g(t)-\tilde g(t)\|_{\hbar,r}^2 \lesssim_r \left(\|h(t)\|_{\hbar,r}^2 + \|\tilde h(t)\|_{\hbar,r}^2\right)\|g(t)-\tilde g(t)\|_r^2 + \eta^2\|h(t)-\tilde h(t)\|_{\hbar,r}^2 + \left(\|g\|_{\hbar,r}^2 + \|\tilde g\|_{\hbar,r}^2\right)\|h(t)-\tilde h(t)\|_r^2$$

We begin by treat the part $\|g(t)-\tilde g(t)\|_r^2$. Using the Gronwall's inequalities,

$$\begin{aligned}\|g(t)-\tilde g(t)\|_r^2 \leq& \|g(0)-\tilde g(0)\|_r^2 e^{C\int_0^t \|h\|_{\hbar,r}^2+\|\tilde h\|_{\hbar,r}^2} \\ &+ \int_0^t 2e^{C\int_s^t \|h\|_{\hbar,r}^2+\|\tilde h\|_{\hbar,r}^2}\left(\eta^2\|h(s)-\tilde h(s)\|_{\hbar,r}^2 + \left(\|g(s)\|_{\hbar,r}^2+\|\tilde g(s)\|_{\hbar,r}^2\right)\|h(s)-\tilde h(s)\|_r^2\right)\mathrm{d}s \\ \leq& C\|h(0)-\tilde h(0)\|_r^2 + C\eta^2\left[\|h-\tilde h\|_{L_t^\infty(\mathcal{H}^r)}^2 + \|h-\tilde h\|_{L_t^2(\mathcal{H}_\hbar^r)}^2\right].\end{aligned}$$

One get now the estimation of $\|g(t)-\tilde g(t)\|_{\hbar,r}^2$:

$$\begin{aligned}\int_0^\infty \|g(t)-\tilde g(t)\|_{\hbar,r}^2\,\mathrm{d}t \lesssim& \int_0^\infty \Big[\left(\|h(t)\|_{\hbar,r}^2+\|\tilde h(t)\|_{\hbar,r}^2\right)\left(\|h(0)-\tilde h(0)\|_r^2 + \eta^2\|h-\tilde h\|_{\hbar,r L_t^2(\mathcal{H}_\hbar^r)\cap L_t^\infty(\mathcal{H}^r)}^2\right) \\ &+\eta^2\|h(t)-\tilde h(t)\|_{\hbar,r}^2 + \left(\|g\|_{\hbar,r}^2+\|\tilde g\|_{\hbar,r}^2\right)\|h(t)-\tilde h(t)\|_r^2\Big]\,\mathrm{d}t \\ \lesssim& \eta^2\left[\|h-\tilde h\|_{L_t^\infty(\mathcal{H}^r)}^2 + \|h-\tilde h\|_{L_t^2(\mathcal{H}_\hbar^r)}^2\right].\end{aligned}$$

We conclude that

$$\|g-\tilde g\|_{L_t^2(\mathcal{H}_\hbar^r)\cap L_t^\infty(\mathcal{H}^r)} \lesssim \|h(0)-\tilde h(0)\|_r + \eta\|h-\tilde h\|_{L_t^2(\mathcal{H}_\hbar^r)\cap L_t^\infty(\mathcal{H}^r)}.$$

This conclude the proof. □

We can now proof Theorem 2: if $\|g_0\| < \kappa_r\eta$, one can perform a Picard's fix point argument in $\mathcal{E}_\hbar(\kappa,\eta,h_0)$. In addition, $M(v)+\sqrt{M}(v)g(t,v)$ is non-negative for any $(t,v)$ by a maximum principle on the equation 1.10.

## 4. Semi-classical (or grazing collision) limit

We recall that for $g, h$ two test functions, and $H := M + \sqrt{M}h$,

$$\mathcal{L}_0 g(v_1) := \left(\nabla - \frac{v_1}{2}\right)\cdot\int B(M,v_1,v_2)\left[\sqrt{M_2}\left(\nabla_1+\frac{v_1}{2}\right)g_1 - \sqrt{M_1}\left(\nabla_2+\frac{v_2}{2}\right)g_2\right]\sqrt{M_2}\,\mathrm{d}v_2$$

$$\begin{aligned}\mathcal{Q}_0(g,h)(v_1) :=& -\left(\nabla - \frac{v_1}{2}\right)\cdot\int B(H,v_1,v_2)\left[h_2\nabla g_1 + g_2\nabla h_1 - h_1\nabla g_2 - g_1\nabla h_2\right]\sqrt{M_2}\,\mathrm{d}v_2 \\ &+\left(\nabla - \frac{v_1}{2}\right)\cdot\int\left(B(H,v_1,v_2)-B(M,v_1,v_2)\right)\left[\sqrt{M_2}\left(\nabla_1+\frac{v_1}{2}\right)g_1 - \sqrt{M_1}\left(\nabla_2+\frac{v_2}{2}\right)g_2\right]\sqrt{M_2}\,\mathrm{d}v_2\end{aligned}$$

$$B(v_1,v_2,H) := c_d\int \frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{|\varepsilon_0(H,k,v_1)|^2}k\otimes k\,\mathrm{d}k$$

**Proposition 4.1.** *We fix the dimension $d \geq 2$. For $f, g$ and $h$ three test functions,*

$$\left|\int f\left(\mathcal{L}_\hbar - \mathcal{L}_0\right)g\right| \lesssim \hbar\|f\|_\hbar\|g\|_5 \tag{4.1}$$

$$\left|\int f\left(\mathcal{Q}_\hbar(g,h) - \mathcal{Q}_0(g,h)\right)\right| \lesssim \hbar\|f\|_\hbar\|g\|_5\|h\|_5(1+\|h\|_5)^4 \tag{4.2}$$

*Proof.* It is sufficient to prove that for any $f, g, h, p$ satisfying

$$\|f\| + \|f\|_\hbar + \|g\|_5 + \|h\|_5 < \infty,\ \ \|p\|_5 \ll 1,$$

there exists an operator $\tilde Q(p,h,g)$ such that

$$\begin{aligned}\int\left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)\left(g_1h_2 - g_1'h_2'\right)\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2|\varepsilon_\hbar(F_p,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 =& \int f_1\tilde Q(p,h,g)_1\,\mathrm{d}v_1 \\ &+ O\left(\hbar\|f\|_\hbar(1+\|p\|_5)^4\|h\|_5\|g\|_5\right)\end{aligned} \tag{4.3}$$

***Step* 1.** In order to identify $\tilde{Q}$, we proceed by duality: for $f, g, h$ and $p$ smooth with compact support,

$$\int \left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)(g_1h_2 + g_2h_1 - g_1'h_2' - g_2'h_1')\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2|\varepsilon_\hbar(F_p,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= \int \left(f_1''\sqrt{M_2} - f_1\sqrt{M_2'}\right)(g_1''h_2 - g_1h_2')\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1)}}{\hbar^2|\varepsilon_\hbar(F_p,-k,v_2)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= -\int \left(\nabla_1 + \frac{v_1}{2}\right)f_1\,(h_2\nabla g_1 - g_1\nabla h_2)\frac{\hat{\mathcal{V}}(k)^2\sqrt{M_2}\delta_{k\cdot(v_2-v_1)}}{2|\varepsilon_0(F_p,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 + O(\hbar)$$
$$= \int f(v_1)\left(\nabla_1 - \frac{v_1}{2}\right)\cdot\int (h_2\nabla g_1 - g_1\nabla h_2)\frac{\hat{\mathcal{V}}(k)^2\sqrt{M_2}\delta_{k\cdot(v_2-v_1)}}{2|\varepsilon_0(F_p,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_2\,\mathrm{d}v_1 + O(\hbar).$$

***Step* 2.** Now we precise the convergence rates in dimension.

$$\int \left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)(g_1h_2 - g_1'h_2')\frac{\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar^2|\varepsilon_\hbar(F_p,k,v_1)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= \int \left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)(h_2\nabla g_1 - g_1\nabla h_2)\cdot\frac{k\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar|\varepsilon_0(F_p,-k,v_2)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 + \mathfrak{R}_\hbar$$

where, denoting $v_{1,s} := v_1 + s\hbar k$ and $v_{2,s} := v_2 - s\hbar k$, we defined $\mathfrak{R}_\hbar$ as

$$\int_0^1 \mathrm{d}s\int\left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)\left(h_{2,s}\nabla^2 g_{1,s} - 2\nabla g_{1,s}\nabla h_{2,s} + g_{1,s}\nabla^2 h_{2,s}\right) : \frac{k^{\otimes 2}\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{|\varepsilon_0(F_p,-k,v_2)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2.$$

Using that

$$\int \left|h_{2,s}\nabla^2 g_{1,s} - 2\nabla g_{1,s}\nabla h_{2,s} + g_{1,s}\nabla^2 h_{2,s}\right|^2|k|^4|\hat{\mathcal{V}}(k)|^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$\lesssim \int |k|^3|\hat{\mathcal{V}}(k)|^2\|g\|_2^2\|h\|^2_{L^\infty(\mathbb{R}\hat{k},H^2(<k>^\perp))}\,\mathrm{d}k \lesssim O(\|g\|_3^2\|h\|_3^2), \tag{4.4}$$

$$\int \left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)^2|\hat{\mathcal{V}}(k)|^2\delta_{k\cdot(v_2-v_1-\hbar k)}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$\lesssim \int |k|^3|\hat{\mathcal{V}}(k)|^2\|g\|_2^2\|h\|^2_{L^\infty(\mathbb{R}\hat{k},H^2(<k>^\perp))}\,\mathrm{d}k \lesssim O(\|g\|_3^2\|h\|_3^2), \tag{4.5}$$

we can apply the Cauchy-Schwartz inequality, and get $\mathfrak{R}_\hbar = O(\|f\|_\hbar\|g\|_3\|h\|_3)$.

Using the change of variable $(v_1, v_2) \mapsto (v_1, v_2')$ and

$$\int \left(f_1\sqrt{M_2} - f_1'\sqrt{M_2'}\right)(h_2\nabla g_1 - g_1\nabla h_2)\cdot\frac{k\hat{\mathcal{V}}(k)^2\delta_{k\cdot(v_2-v_1-\hbar k)}}{\hbar|\varepsilon_0(F_p,-k,v_2)|^2}\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= \int f_1\left(\frac{\sqrt{M_2''}\,(h_2''\nabla g_1 - g_1\nabla h_2'')}{|\varepsilon_0(F_p,-k,v_2'')|^2} - \frac{\sqrt{M_2'}\,(h_2\nabla g_1'' + g_2\nabla h_1'' - g_1''\nabla h_2 - h_1''\nabla g_2)}{|\varepsilon_0(F_p,-k,v_2)|^2}\right)$$
$$\cdot\,\hbar^{-1}k\delta_{k\cdot(v_2-v_1)}\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2$$
$$= \int f_1 k\cdot\left(\nabla_2 + \nabla_1 + \tfrac{v_2}{2}\right)\frac{\sqrt{M_2}\,(h_2\nabla g_1 - g_1\nabla h_2)}{|\varepsilon_0(F_p,-k,v_2)|^2}\cdot k\delta_{k\cdot(v_2-v_1)}\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2 + O(\hbar\|f\|_\hbar\|g\|_4\|h\|_4\|p\|_4),$$

We use that

$$\int_0^1\int |f_1|^2\langle\hat{k}\cdot v_{2,s}\rangle^4 M_{2,s}\delta_{k\cdot(v_2-v_1)}|k|^3\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}s \lesssim \int |f_1|^2\exp\left(-\tfrac{(v_1\cdot\hat{k})^2}{4}\right)|k|^3\hat{\mathcal{V}}(k)^2\,\mathrm{d}k\,\mathrm{d}v_1$$
$$\lesssim \left\|\frac{f}{\langle v\rangle^{\frac{1}{2}}}\right\|^2 \lesssim \|f\|_\hbar^2.$$

The other term of the product is estimated using the same argument than in (4.4).

This conclude the proof. □

*Proof of Theorem 4.* First, as $\|g_0\|_r < \eta$ and $\hbar < \hbar_0$, one can applied Theorem 2, and for any $\hbar < \hbar_0$, there exists $g_\hbar$ solution of (QLB$_\hbar$). In addition one have the following bound:

$$\sup_{t\geq 0}\|g_\hbar(t)\|_r^2 + \int_0^\infty \|g_\hbar(t)\|_{\hbar,r}^2\,\mathrm{d}t \lesssim \eta^2.$$

We deduce that the family $(g_\hbar)_{\hbar<\hbar_0}$ is weakly compact in $\mathcal{C}_b^0(\mathbb{R}^+,\mathcal{H}^r)\cap L^2(\mathcal{H}_\hbar^r)$ for all $\hbar>0$. Hence, up to the extraction of a subsequence $\hbar'$, $(g_{\hbar'})$ converges weakly to some function $g_\infty\in\mathcal{C}_b^0(\mathbb{R}^+,\mathcal{H}^r)\cap L^2(\mathcal{H}_0^r)$.

We need to identify the limit. In [DW23], the authors show the existence of $g_\infty\in\mathcal{C}_b^0(\mathbb{R}^+,\mathcal{H}^r)\cap L^2(\mathcal{H}_0^r)$, the unique solution of (CLB)-equation, with initial data $g_\infty(t=0):=g_0$[1].

We perform again an energy estimation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g_\hbar(t)-g_\infty(t)\|^2 =& -2\int(g_\hbar(t)-g_\infty(t))\mathcal{L}_\hbar(g_\hbar(t)-g_\infty(t))\\ &-2\int(g_\hbar(t)-g_\infty(t))(\mathcal{L}_0g_\infty(t)-\mathcal{L}_\hbar g_\infty(t))\\ &+2\int(g_\hbar(t)-g_\infty(t))\left(\mathcal{Q}_\hbar(g_\hbar(t),g_\hbar(t))-\mathcal{Q}_\hbar(g_\infty(t),g_\infty(t))\right)\\ &+2\int(g_\hbar(t)-g_\infty(t))\left(\mathcal{Q}_\hbar(g_\infty(t),g_\infty(t))-\mathcal{Q}_\infty(g_\infty(t),g_\infty(t))\right)\end{aligned}$$

Using Proposition 3.7, 3.10 and 4.1, and using the same strategy than in the same way than in the proof of Proposition 3.12,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g_\hbar(t)-g_\infty(t)\|^2+(1-\eta)\|g_\hbar(t)-g_\infty(t)\|_\hbar^2\lesssim\hbar\|g_\hbar(t)-g_\infty(t)\|_\hbar\|g_\infty\|_5\left(1+\|g_\infty\|_5\right)^4.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g_\hbar(t)-g_\infty(t)\|^2+(1-2\eta)\|g_\hbar(t)-g_\infty(t)\|_\hbar^2\lesssim\frac{\hbar^2}{\eta}\|g_\infty\|_5^2\left(1+\|g_\infty\|_5\right)^8.$$

As $r\geq 5$, we can applied the Gronwall inequality, and obtain

$$\|g_\hbar(t)-g_\infty(t)\|\leq\hbar t\|g_\infty\|_{L^2_t(\mathcal{H}^5_\hbar)\cap L^\infty_t(\mathcal{H}^5)}(1+\|g_\infty\|_{L^2_t(\mathcal{H}^5_\hbar)\cap L^\infty_t(\mathcal{H}^5)})^4$$

This conclude the proof. □

## 5. Short time existence theorem

In order to obtain short time result for arbitrary large initial datum, one use the trivial bound

$$\|f\|_\hbar\lesssim\frac{1}{\hbar}\|f\|$$

Then, we can prove the following estimation by using Proposition 3.11,

**Proposition 5.1.** *We fix the dimension $d\geq 2$. Denoting $\tilde r:=\max(5,r)$, for $f,g,h$ three tests functions with*

$$|\varepsilon_\hbar(M+\sqrt{M}h,k,v)|\geq\frac{1}{C_h},$$

*we have the following bounds*

$$\left|\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\mathcal{Q}_\hbar(g,h)\right|\lesssim\frac{1}{\hbar^2}\left(1+\|h\|_{\tilde r}\right)^{r+1}\|g\|_{\tilde r}\|h\|_{\tilde r}\|f\|_{\tilde r}$$

$$\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\mathcal{L}_\hbar g\lesssim\frac{1}{\hbar^2}\|g\|_{\tilde r}\|\,\|f\|_{\tilde r}$$

$$\begin{aligned}&\left|\int\left(-\nabla-\frac{v}{2}\right)^r\left(\nabla-\frac{v}{2}\right)^r f\left[\mathcal{Q}_\hbar(g,h)-\mathcal{Q}_\hbar(\tilde g,\tilde h)\right]\right|\\ &\quad\lesssim\frac{1}{\hbar^2}C_h^{2+r}\left(1+\|h\|_{\tilde r}+\|\tilde h\|_{\tilde r}\right)^{r+1}\left(\|g-\tilde g\|_{\tilde r}\left(\|h\|_{\tilde r}+\|\tilde h\|_{\tilde r}\right)+\left(\|g\|_{\tilde r}+\|\tilde g\|_{\tilde r}\right)\|h-\tilde h\|_{\tilde r}\right)\|f\|_{\tilde r}\end{aligned}$$

Fix $g_0>0$ a probability density and a constant $C_{g_0}$ with the following bounds

$$|\varepsilon_\hbar(M+\sqrt{M}g_0,v,k)|\geq\frac{1}{C_{g_0}},\quad\|g_0\|_r\leq C_{g_0}. \tag{5.1}$$

We define the set

$$\tilde{\mathcal{E}}_r(T,g_0,C_{g_0})=\left\{h\in\mathcal{C}([0,T],\mathcal{H}^r),\ h(t=0)=g_0,\ \|h-g_0\|_{L^\infty([0,T],\mathcal{H}_r)}\leq\frac{1}{2\|\mathcal{V}\|_{L^1}}C_{g_0}\right\}$$

[1]In reality, their results is weaker, but one can easily adapt their proof.

**Proposition 5.2.** *We fix the Planck constant $\hbar$ and an initial data $g_0$ (associated with the bound $C_{g_0}$). We define $\mathcal{F}_{\hbar,T}$ the application which associate to $\tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$ the solution $g$ on $[0,T]$ of the equation*

$$\begin{cases} \partial_t g(t) + \mathcal{L}_\hbar g(t) = \mathcal{Q}_\hbar(g(t), h(t)) \\ g(t=0) = h(t=0). \end{cases} \tag{5.2}$$

*Then for $r > 5$, there exists a constant $c_0$ independant of $\hbar, C_{g_0}$ such that for any time $T < \frac{c_0 \hbar^2}{C_{g_0}^{2r+4}}$,*

- *for all $h \in \tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$, the dielectric constant is bounded from below:*
$$|\varepsilon_\hbar(M + \sqrt{M} h(t), v, k)| \geq \frac{1}{2C_{g_0}},$$
*hence, the application $\mathcal{F}_{\hbar,T}$ is well defined on $\tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$,*
- *the set $\tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$ is stable under the action of $\mathcal{F}_\hbar$,*
- *the application $\mathcal{F}_\hbar$ is contractive on $\tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$.*

*Proof.* We begin by the lower bound of the dielectric constant:

$$\begin{aligned} &|\varepsilon_\hbar(M + \sqrt{M} h(t), v, k)| \\ &\qquad \geq |\varepsilon_\hbar(M + \sqrt{M} g_0, v, k)| - \hat{\mathcal{V}}(k) \left| \int_1^2 \mathrm{d}s \int \frac{\hat{k} \cdot \nabla \left[\sqrt{M}(h(t) - g_0)\right](v_* - s\hbar k)}{\hat{k} \cdot (v - v_*) - i0} \mathrm{d}v_* \right| \geq \frac{1}{2C_{g_0}}. \end{aligned}$$

The proof looks like the proof of Proposition 3.12.

Fix $h, h' \in \tilde{\mathcal{E}}_r(T, g_0, C_{g_0})$, and denote $g := \mathcal{F}_\hbar(h)$, $g' := \mathcal{F}_\hbar(h')$. For $\eta \in (0,1)$, we define $T_\eta$ as the supremum of

$$\left\{ \tau \mid \forall \tau' \in (0,\tau),\ \|g(\tau') - g_0\|_r, \|g'(\tau') - g_0\|_r \leq \frac{C_{g_0}}{2}, \|g(\tau') - g'(\tau')\|_r \leq \frac{\|h - h'\|_{L^\infty([0,T_\eta],\mathcal{H}^r)}}{2} \right\}.$$

For $r' \leq r$ and $t \leq T$

$$\begin{aligned} \frac{d}{dt} \left\| \left(\nabla - \frac{v}{2}\right)^{r'} (g(t) - g_0) \right\|^2 &= -2 \int \left(-\nabla - \frac{v}{2}\right)^{r'} \left(\nabla - \frac{v}{2}\right)^{r'} (g(t) - g_0) \mathcal{L}_\hbar g(t) \\ &\quad - 2 \int \left(-\nabla - \frac{v}{2}\right)^{r'} \left(\nabla - \frac{v}{2}\right)^{r'} (g(t) - g_0) \mathcal{Q}_\hbar(g(t), h(t)) \\ &\lesssim \frac{C_{g_0}^{2r'+5}}{\hbar^2} \|g(t) - g_0\|_{r'}. \end{aligned}$$

Summing on $r'$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g(t) - g_0\|_r^2 \lesssim \frac{C_{g_0}^{2r+5}}{\hbar^2} \|g(t) - g_0\|_r,$$

and the same inequality hold for $g'$. In the same way, there exists some constant $C$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|g(t) - g'(t)\|_r^2 &\leq C \frac{C_{g_0}^{2r+4}}{\hbar^2} \|g(t) - g'(t)\|_r (\|g(t) - g'(t)\|_r + \|h(t) - h'(t)\|_r) \\ &\leq \frac{3CC_{g_0}^{2r+4}}{2\hbar^2} \left( \|g(t) - g'(t)\|_r^2 + \|h - h'\|_{L^\infty([0,T_\eta],\mathcal{H}^r)}^2 \right). \end{aligned}$$

Using the Gronwall lemma, one have

$$\begin{aligned} \varpi_1(t) &:= \max\left(\|g(t) - g_0\|_r, \|g'(t) - g_0\|_r\right) \lesssim \frac{C_{g_0}^{2r+5}}{\hbar^2} t, \\ \varpi_2(t) &:= \|g(t) - g'(t)\|_r \lesssim \left[ \exp\left( \frac{3CC_{g_0}^{2r+4} \eta}{2\hbar^2} t \right) - 1 \right]^{\frac{1}{2}} \|h - h'\|_{L^\infty([0,T_\eta],\mathcal{H}^r)}. \end{aligned}$$

By continuity of $\varpi_1$ and $\varpi_2$, at $T_\eta$, either $\varpi_1(T_\eta) = C_{g_0}/2$ or $\varpi_2(T_\eta) = \|h - h'\|_{L^\infty([0,T_\eta],\mathcal{H}^r)}/2$. Hence, there exists a constant $c_0$ such that $T_\eta > \frac{c_0 \hbar^2}{C_{g_0}^{2r+4}}$. ☐

We conclude by a Picard fix-point argument.

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