# TD 5: Weak topology

**Exercise** 1 (Properties of weakly convergent sequences). Let $X$ be a normed vector space.

1. Let $(u_n)_n$ be a weakly convergent sequence in $X$. Justify that $(u_n)$ is bounded and that the weak limit $u$ of $(u_n)_n$ satisfies $\|u\| \leq \liminf_{n\to+\infty} \|u_n\|$.
2. Suppose that the sequence $(\varphi_n)_n$ in $X^*$ is converging strongly to some $\varphi \in X^*$. Show that for any sequence $(u_n)_n$ in $X$ that converges weakly to $u \in X$, then the sequence $(\varphi_n(u_n))_n$ converges strongly to $\varphi(u)$.
3. Assume that $X$ is a Hilbert space. Let $(u_n)_n$ be a sequence in $X$ that converges weakly to $u \in X$ and such that $(\|u_n\|)_n$ converges to $\|u\|$. Prove that $(u_n)_n$ converges strongly to $u$.

**Exercise** 2 (Examples of weakly convergent sequences).

1. Let $H$ be a separable Hilbert space and $(e_n)_n$ be a Hilbert basis of $H$. Prove that $(e_n)_n$ converges weakly to 0 but not strongly.
2. Let $K \subset \mathbb{R}^d$ be a compact set. Show that weak convergence in $C(K)$ is equivalent to bounded pointwise convergence.
3. Let $\Omega \subset \mathbb{R}^d$ and $(u_n)_n$, $(v_n)_n$ be two sequences in $L^2(\Omega)$ such that $(u_n)_n$ converges weakly and $(v_n)_n$ strongly. Show that the sequence $(u_n v_n)_n$ converges weakly in $L^1(\Omega)$. What happens if the two sequences converge weakly ?

**Exercise** 3 (Weak topology). Let $X$ be a topological vector space. Show that $X$, endowed with the weak topology, is a locally convex topological vector space.

**Exercise** 4. Let $E$ be a Banach space.

1. Show that if $E$ is finite-dimensional, then the weak topology $\sigma(E, E^*)$ and the strong topology coincide.
2. We assume that $E$ is infinite-dimensional.
   (a) Show that every weak open subset of $E$ contains a straight line.
   (b) Deduce that $B = \{x \in E : \|x\| < 1\}$ is not open for the weak topology.
   (c) Let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. What is the weak closure of $S$ ?

**Exercise** 5. Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We introduce the canonical family of sequences $e^k$ in $\ell^p(\mathbb{N})$, for which every term is zero, except the $k^{\text{th}}$ which is 1. We also consider the map

$$\begin{array}{rccc} J_p : & \ell^q(\mathbb{N}) & \to & (\ell^p(\mathbb{N}))^* \\ & (a_n)_n & \mapsto & \left( (x_n)_n \mapsto \displaystyle\sum_{n=0}^{+\infty} a_n x_n \right) \end{array}$$

1. When $p \in [1, \infty)$, show that $J_p$ is a surjective isometry.
2. Show that $J_\infty$ is a non-surjective isometry.

3. When $p \in (1, \infty)$, prove that the sequence $(e^k)_k$ converges weakly but not strongly in $\ell^p(\mathbb{N})$ towards the null sequence.
4. Still assuming that $p \in (1, \infty)$, we consider the following subset of $\ell^p(\mathbb{N})$:

$$E = \left\{ e^n + n e^m : n, m \in \mathbb{N},\ m > n \right\}.$$

   (a) Show that $E$ is closed for the strong topology in $\ell^p(\mathbb{N})$.
   (b) Show that 0 is in the weak closure of $E$.
   (c) Show that a sequence of $E$ cannot converge weakly towards 0.
   (d) Deduce that the weak topology on $\ell^p$ is not metrizable.

**Exercise** 6.

1. (Mazur's lemma) Let $E$ be a Banach space and $(u_n)_n$ be a sequence in $E$ weakly converging to $u_\infty \in E$. Show that $u_\infty$ is a strong limit of finite convex combinations of the $u_n$.
2. (Banach-Sacks' property) Show that if $E$ is in addition a Hilbert space, we can extract a subsequence converging to $u_\infty$ strongly in the sens of Cesàro.

**Exercise** 7 (Schur's property for $\ell^1(\mathbb{N})$).

1. Recall why weak and strong topologies always differ in an infinite dimensional norm vector space.

The aim is to prove that a sequence of $\ell^1(\mathbb{N})$ converges weakly if and only if it converges strongly. Take $(u^n)_n$ a sequence in $\ell^1(\mathbb{N})$ weakly converging to 0.

2. Show that for all $k$, $\lim_{n\to\infty} u_k^n \to 0$.
3. Show that if $u_n \nrightarrow 0$ in $\ell^1(\mathbb{N})$, one can additionally assume that $\|u^n\|_{\ell^1} = 1$.
4. Define via a recursive argument two increasing sequences of $\mathbb{N}$, $(a_k)_k$ and $(n_k)_k$, such that

$$\forall k \geq 0, \qquad \sum_{j=a_k}^{a_{k+1}-1} |u_j^{n_k}| \geq \frac{3}{4}.$$

5. Show that there exists $v \in \ell^\infty(\mathbb{N})$ such that $(v, u^{n_k})_{\ell^2} \geq \frac{1}{2}$ for all $k$. Conclude.