TD 2: L^p compactness and Banach spaces

Exercise 1 (F. Riesz's theorem). Let E be a normed vector space.

1. Prove that if M is a closed subspace of E, with $M \neq E$, then for all $\varepsilon > 0$, there exists $u \in E$ of norm $\|u\| = 1$ such that $d(u, M) \geq 1 - \varepsilon$.
2. Deduce that if E is infinite-dimensional, then its unit ball $\mathcal{B}$ is not compact, with

$$\mathcal{B} = \big\{x \in E : \|x\| \leq 1\big\}.$$

Exercise 2 (Norm on the quotient space). Let E be a Banach space and M be a closed vector subspace of E. Let us consider $N : E/M \to \mathbb{R}$ defined by

$$N(\xi) = \inf_{\xi = \overline{x}} \|x\|.$$

Prove that N defines a norm on E/M, and that E/M is a Banach space.
Hint: Prove that if $(u_n)_n$ is a Cauchy sequence, then one can extract a subsequence $(n_k)_k$ such that

$$\forall k \geq 0, \quad \|u_{n_{k+1}} - u_{n_k}\| \leq \frac{1}{2^k}.$$

Exercise 3 (Characterization of equi-integrability). Let (X, μ) be a measured space and $\mathcal{F} \subset L^1(X, \mu)$ being bounded. Prove that the following assertions are equivalent:

1. $\mathcal{F}$ is equi-integrable,
2. For all $\varepsilon > 0$, there exists some $\eta > 0$ such that for any measurable set A,

$$\mu(A) < \eta \Rightarrow \sup_{u \in \mathcal{F}} \int_A |u| \, \mathrm{d}\mu < \varepsilon.$$

3. There exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to \infty} \Phi(x)/x = \infty$ and

$$\sup_{u \in \mathcal{F}} \int_X \Phi(|u|) \, \mathrm{d}\mu < \infty.$$

Hint: to show 2. $\Rightarrow$ 3., consider the sequence $(M_n)_n$ such that

$$\sup_{u \in \mathcal{F}} \int_X |u| \mathbb{1}_{|u| > M_n} \, \mathrm{d}\mu < 2^{-n}.$$

Exercise 4 (Vitali's convergence theorem). We consider $(X, \mathcal{A}, \mu)$ a σ-finite measured space. Let $1 \leq p < +\infty$ and $(u_n)_n$ be a sequence in $L^p(X)$. Assume that

1. $(u_n)_n$ is a Cauchy sequence in measure, meaning that for all $\varepsilon > 0$, there exists $n_0 \geq 0$ such that

$$\forall m, n \geq n_0, \quad \mu\big(|u_n - u_m| \geq \varepsilon\big) < \varepsilon.$$

2. $(u_n)_n$ is equi-integrable in $L^p(X)$,
3. for all $\varepsilon > 0$, there exists a measurable set Γ of finite measure such that

$$\forall n \geq 0, \quad \|u_n \mathbb{1}_{X \setminus \Gamma}\|_{L^p(X)} \leq \varepsilon.$$

Prove that $(u_n)_n$ is a Cauchy sequence in $L^p(X)$ (and therefore converges in this space).

Exercise 5 (Obstructions to strong convergence). The purpose of this exercise is to present three obstructions to strong convergence in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$. In the following, $\varphi \in C_c^\infty(\mathbb{R}^d)$ denotes a compactly supported smooth function being not identically equal to zero.

1. (Loss of mass) Let ν be a vector of norm 1. Prove that the sequence $(\varphi(\cdot - n\nu))_n$ does not converge in $L^2(\mathbb{R}^d)$.
2. (Concentration) Prove that the sequence $(n^{d/2}\varphi(n\,\cdot))_n$ does not converge in $L^2(\mathbb{R}^d)$.
3. (Oscillations) We now consider $w \in L^2(\mathbb{T}^d)$ a non-constant function. Prove that the sequence $(w(n\,\cdot))_n$ does not converge in $L^2(\mathbb{T}^d)$.

Exercise 6 (Averaging lemma). Let $u \in \mathscr{S}(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ be a Schwartz function. For any function $\phi \in C_c^\infty(\mathbb{R}^d)$, we consider the moment

$$\rho_\phi(x) := \int_{\mathbb{R}^d} \phi(v) u(x,v) \, \mathrm{d}v.$$

1. Let us define $\hat{u}(\xi, v)$ as the Fourier transform of the function u with respect to the space variable $x \in \mathbb{R}^d$. Considering the function $w := (1 + v \cdot \nabla_x)u$, show that for all $\xi \in \mathbb{R}^d$,

$$|\hat{\rho}_\phi(\xi)|^2 \leq \left(\int_{\mathbb{R}^d} |\hat{w}|^2(\xi, v) \, \mathrm{d}v \right) \left(\int_{\mathbb{R}^d} \frac{\phi^2(v) \, \mathrm{d}v}{1 + |v \cdot \xi|^2} \right).$$

2. Deduce that

$$\|\rho_\phi\|^2_{H^{1/2}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^{1/2} |\hat{\rho}_\phi|^2(\xi) \, \mathrm{d}\xi \leq C_\phi \big(\|u\|^2_{L^2(\mathbb{R}^{2d})} + \|v \cdot \nabla_x u\|^2_{L^2(\mathbb{R}^{2d})} \big),$$

where the constant $C_\phi > 0$ only depends on the function ϕ.