TD 11: Sobolev spaces

Exercise 1 (Warming).

1. Show that $u(x) = |x|$ belongs to $W^{1,2}(-1,1)$ but not to $W^{2,2}(-1,1)$.
2. Check that $v(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$ belongs to $L^2(\mathbb{R})$ but not to $W^{1,2}(\mathbb{R})$.
3. Show that $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$.
   Hint: Consider a function of the form $x \mapsto \chi(|x|)\,|\log|x||^{1/3}$.

Exercise 2 (Optimality in the Sobolev embeddings). Let $1 \le p < d$ and $\alpha \in [1,\infty]$. By using a homogeneity argument, show that if there exists a continuous injection $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\alpha(\mathbb{R}^d)$, then necessarily $p \le \alpha \le \frac{dp}{d-p}$.

Exercise 3 (Some properties of $H^s(\mathbb{R}^d)$).

1. Show that $H^{s_1}(\mathbb{R}^d)$ embeds continuously into $H^{s_2}(\mathbb{R}^d)$ for $s_1 \ge s_2$.
2. Check that $\delta_0 \in H^s(\mathbb{R}^d)$ for $s < -d/2$.
3. (a) Prove that if $s > d/2$, the space $H^s(\mathbb{R}^d)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^d)$, the space of continuous functions u on $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
   (b) State an analogous result in the case where $s > d/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.
   (c) Let $U \subset \mathbb{R}^d$ be open. Deduce from the above question that $\bigcap_{s\in\mathbb{R}} H^s_{loc}(U) = C^\infty(U)$, where we set
   $$H^s_{loc}(U) = \big\{u \in L^2(U) : \forall \varphi \in \mathcal{D}(U),\ \varphi u \in H^s(\mathbb{R}^d)\big\}.$$
4. Let us now consider $s \in (d/2, d/2+1)$.
   (a) Show that for all $\alpha \in [0,1]$ and all $x, y, \xi \in \mathbb{R}^d$:
   $$\left|e^{ix\cdot\xi} - e^{iy\cdot\xi}\right| \le 2^{1-\alpha}|x-y|^\alpha|\xi|^\alpha.$$
   (b) Deduce that for all $\alpha \in (0, s-d/2)$, there exists a constant $C(\alpha) > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,
   $$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C(\alpha)\|u\|_{H^s}.$$
   (c) Conclude that $H^s(\mathbb{R}^d)$ embeds continuously to $C^\alpha(\mathbb{R}^d)$.
5. Assuming that s belongs to $[0, d/2]$, the purpose is now to prove that $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, where $p = 2d/(d-2s)$. To that end, let us recall that for all $u \in L^p(\mathbb{R}^n)$,
   $$\|u\|^p_{L^p} = \int_0^\infty p\lambda^{p-1}\big|\{|u| > \lambda\}\big|\,\mathrm{d}\lambda.$$
   Considering $u \in \mathcal{S}(\mathbb{R}^d)$ and $A_\lambda > 0$, we set $u_{1,\lambda} = \mathcal{F}^{-1}(\mathbb{1}_{|\xi|<A_\lambda}\hat{u})$ and $u_{2,\lambda} = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\ge A_\lambda}\hat{u})$.

(a) Prove that

$$\forall x \in \mathbb{R}^d, \quad |u_{1,\lambda}(x)| \leq C A_\lambda^{(2d-s)/2} \|u\|_{H^s}.$$

Deduce that there exists some A_λ such that $|\{|u_{1,\lambda}| > \lambda/2\}| = 0$.

(b) Show that for this choice of A_λ,

$$\|u\|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \|u_{2,\lambda}\|_{L^2}^2 \, \mathrm{d}\lambda.$$

(c) Conclude.

Exercise 4 (Trace on an hyperplane). Let us consider the function

$$\gamma_0 : \varphi(x', x_d) \in C_0^\infty(\mathbb{R}^d) \mapsto \varphi(x', x_d = 0) \in C_0^\infty(\mathbb{R}^{d-1}).$$

Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
Hint: For all $\varphi \in C_0^\infty(\mathbb{R}^d)$, begin by computing the Fourier transform of the function $\gamma_0 \phi$.

Exercise 5 (An estimate). Let $0 < \alpha < 1$ and $p > 1$ be positive real numbers. Show that there exists a positive constant $C_{\alpha,p} > 0$ such that for all $u \in C_0^\infty(\mathbb{R})$,

$$\left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{|u(x) - u(y)|}{|x-y|^\alpha} \right)^p \frac{\mathrm{d}x \mathrm{d}y}{|x-y|^d} \right)^{1/p} \leq C_{\alpha,p} \|u\|_{L^p(\mathbb{R}^d)}^{1-\alpha} \|\nabla u\|_{L^p(\mathbb{R}^d)}^\alpha.$$

Hint: Consider the two regions $\{|x-y| > R\}$ and $\{|x-y| \leq R\}$, where $R > 0$ is to be chosen.

Exercise 6 (Composition). Let U and U' be two open subset of $\mathbb{R}^d$.

1. Let $H : U' \to U$ be a C^1-diffeomorphism such that the Jacobian $\mathrm{Jac}(H)$ and $\mathrm{Jac}(H^{-1})$ belong to L^∞. Prove that for all $u \in W^{1,p}(\Omega)$, we have $u \circ H \in W^{1,p}(\Omega')$ and that for all $1 \leq i \leq d$,

$$\partial_{y_i}(u \circ H) = \sum_{j=1}^n (\partial_{x_j} u \circ H) \partial_{y_i} H_j.$$

2. Let us now consider a function $G \in C_b^1(\mathbb{R})$ satisfying $G(0) = 0$. Show that for all $u \in W^{1,p}(U)$, we have $G \circ u \in W^{1,p}(U)$ and that for all $1 \leq j \leq n$,

$$\partial_{x_j}(G \circ u) = (G' \circ u) \partial_{x_j} u.$$

3. Do we need to assume that G' is bounded when $d = 1$?