TD 10: Tempered distribution

Exercise 1.

1. Let $A \subset \mathbb{R}^d$ be a Borel of finite measure. Show that $\mathcal{F}(\mathbb{1}_A)$ belongs to $L^2(\mathbb{R}^d)$ but not to $L^1(\mathbb{R}^d)$.
2. Does it exist two functions $f, g \in \mathcal{S}(\mathbb{R})$ such that $f * g = 0$? What happens if in addition f and g have compact supports ?

Exercise 2. Prove that the following distributions are tempered and compute their Fourier transform:

1. δ_0 in $\mathbb{R}^d$,
2. $e^{-\frac{|x|^2}{2\sigma}}$ in $\mathbb{R}$ with $\sigma > 0$,
3. 1,
4. H (Heaviside),
5. $\mathrm{p.v.}(1/x)$,
6. $|x|$ in $\mathbb{R}$.

Exercise 3.

1. If $d \geq 3$, show that $u_0(x) = \left(-d(d-2)\mathrm{Vol}(B(0,1))\|x\|^{d-2}\right)^{-1}$ is a fundamental solution for the Laplacian, *i.e.* $\Delta u_0 = \delta_0$ in the sense of distributions.
2. Give a solution of $\Delta u = f$ in the sense of distributions for f in $\mathcal{D}'(\mathbb{R}^d)$ with compact support.
3. What can you say about the regularity of u if f is a function in $\mathcal{S}(\mathbb{R}^d)$?
4. Consider the linear PDE $u - \Delta u = f$ for $f \in \mathcal{S}(\mathbb{R}^d)$. Express a solution in $\mathcal{S}(\mathbb{R}^d)$ in terms of the Bessel kernel $B = \mathcal{F}^{-1}((1+|\xi|^2)^{-1})$.

Exercise 4. Let $k > 0$ and $T \in \mathcal{S}'(\mathbb{R})$ such that $T^{[4]} + kT \in L^2(\mathbb{R})$. Show that for every $j \in \{0, \cdots, 4\}$, $T^{[j]} \in L^2(\mathbb{R})$.

Exercise 5. We investigate the solutions $T \in \mathcal{S}'(\mathbb{R}^4)$ with support in $\mathbb{R}_+ \times \mathbb{R}^3$ of the wave equation

$$\partial_{tt} T - \Delta T = \delta_{(t,x)=(0,0)}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3.$$

1. Let $\mathcal{F}$ be the partial Fourier transform with respect to x and $\tilde{T} = \mathcal{F}T$. Find an ODE of which $\tilde{T}$ is solution. We denote in the following (E) this equation.
2. Solve this equation with the ansatz
$$\tilde{T}(t,\xi) = H(t)U(t,\xi),$$
where U is solution of the homogenous equation associated with (E).
3. We denote by $\mathrm{d}\sigma_R$ the measure on the sphere of radius R and center 0:
$$\langle \mathrm{d}\sigma_R, \varphi \rangle = \int_{\mathbb{S}(0,R)} \varphi(x)\,\mathrm{d}\sigma_R(x)$$
Show that:
$$\forall \xi \in \mathbb{R}^d, \quad \mathcal{F}\left(\frac{\mathrm{d}\sigma_R}{4\pi R^2}\right)(\xi) = \frac{\sin(R|\xi|)}{R|\xi|}.$$

4. Deduce that for $\varphi \in \mathcal{S}(\mathbb{R}^4)$,

$$\langle T, \varphi \rangle = \int_0^\infty \frac{1}{4\pi t} \int_{\mathbb{S}(0,|t|)} \varphi(t,x)\,\mathrm{d}\sigma_t(x)\,\mathrm{d}t.$$

5. What is the support of T?

Exercise 6. We consider the Schrödinger equation on $\mathbb{R}_t \times \mathbb{R}^d$

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u_{t=0} = u_0. \end{cases} \tag{1}$$

1. For $u_0 \in \mathcal{S}(\mathbb{R}^d)$, solve the equation (1) in $C^0(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$.
2. Justify that the Fourier transform of the function $e^{it|\xi|^2}$ is well defined.
3. Show that for $\alpha \in \mathbb{C}$ with positive real part,

$$\mathcal{F}^{-1}(e^{\alpha|\xi|^2}) = \frac{1}{(-4\alpha\pi)^{d/2}} e^{\frac{|x|^2}{4\alpha}}.$$

4. Check that also holds in $\mathcal{S}'(\mathbb{R}^d)$ when $\alpha \in i\mathbb{R}$.
5. Deduce that there exists a constant $C > 0$ such that for all $t > 0$,

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \frac{C}{t^{d/2}} \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$